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Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution

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ABSTRACT
Inference based on popular maximum-likelihood estimators (MLEs) method often provide bias estimates of order $\mathcal{O}(n^{-1})$. Such bias may significantly affect the accuracy of estimates. This observation motivates us to adopt some bias-corrected technique to reduce the bias of the MLE from order $\mathcal{O}(n^{-1})$ to order $\mathcal{O}(n^{-2})$. In this paper, we consider the unit-gamma distribution which has some properties similar to the Beta distribution. This distribution is obtained by transforming a Gamma random variable but it has not been widely explored in the literature. We adopt a "corrective" approach to derive second-order bias corrections of the MLEs of its parameters. Additionally, we also consider the parametric Bootstrap bias correction. Monte Carlo simulations are conducted to investigate the performance of proposed estimators. Our results revels the bias corrections improve the accuracy of estimates. Finally, two real data examples are discussed to illustrate the applicability of the unit-Gamma distribution.

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MATHEMATICS SUBJECT CLASSIFICATION

1. Introduction

Let X be a non negative random variable which follows a Gamma distribution with probability density function (PDF) given by

$$f(x \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x} \tag{1}$$

where $\Gamma(u) = \int_0^{\infty} u^{\alpha-1} e^{-u} \, \mathrm{d}u$ is the complete gamma function, $\alpha > 0$ is the shape parameter and $\beta > 0$ is the rate parameter. By considering the transformation:

$$Y = e^{-X} \tag{2}$$

Grassia (1977) derived a new distribution which was called by Ratnaparkhl and Mosimann (1990) the unit-Gamma (UG) distribution, since its support is on the unit interval. The PDF of Y is defined as:

$$f(y \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\beta-1} (-\log y)^{\alpha-1} \tag{3}$$

where $0 < y < 1$ and $\alpha > 0$ and $\beta > 0$ are the shape parameters. The PDF (3) includes the bell shaped ($\alpha > 1$ and $\beta > 1$), the J-shaped (inverted and reversed), triangular shaped and

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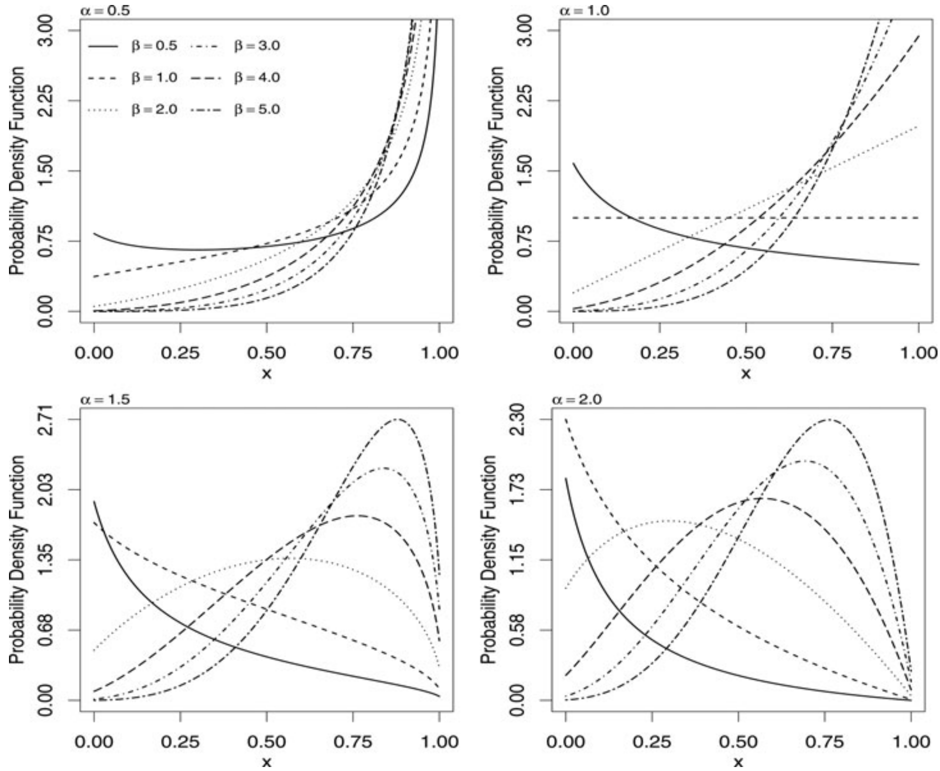

Figure 1. Unit-gamma PDF considering different values for α and β.

U-shaped curves (see Grassia 1977). Figure 1 display some forms of the PDF of UG distribution considering different values for α and β.

A brief historical account of the UG distribution with its variants has been studied by Grassia (1977). He considered the use of the UG distribution as a mixing distribution for the parameter of the binomial distribution in some cases instead of Beta distribution with the Poisson distribution resulting in a more convenient form of the compound distribution. The UG distribution has been found useful in applications like inoculation approach to estimate bacteria or virus density in dilution assays with host variability to infection and for deriving other statistical distributions (see Grassia 1977; Ratnaparkhl and Mosimann 1990).

To the best of our knowledge, this distribution has not been widely explored in the statistical literature. Tadikamalla (1981) in his discussion paper pointed out that this distribution can be used as an alternative for Beta and Johnson S_B distributions. He also investigated some of its properties. Ratnaparkhl and Mosimann (1990) studied the logarithmic and Tukey's lambda-type transformation on the unit-Gamma distribution. Although not well known, the UG distribution can be a potential model to be used as an alternative to the classical Beta distribution.

Parameter estimation is of utmost importance for any probability distribution. Among all the estimation methods, the most frequently used method is the maximum-likelihood (Pawitan 2001; Millar 2011) method. Its underlying motivation is simple and intuitive. For example, they are asymptotically unbiased, consistent, and asymptotically normally distributed. However, most of these properties essentially rely on the large sample size condition, and hence properties like unbiasedness, may not be applicable for small and moderate sample sizes (Kay 1995). As this method gives biased estimates, researchers strive to develop nearly unbiased estimators for the parameters of several distributions. Readers may refer to Cordeiro et al. (1997), Cribari-Neto and Vasconcellos (2002), Saha and Paul (2005), Lemonte et al. (2007), Giles and Feng (2009), Lagos-Àlvarez et al. (2011), Lemonte (2011), Giles (2012a), Giles (2012b), Schwartz et al. (2013), Giles et al. (2013), Teimouri and Nadarajah (2013), Ling and Giles (2014), Zhang and Liu (2015), Singh et al. (2015), Teimouri and Nadarajah (2016), Reath (2016), Schwartz and Giles (2016), Wang and Wang (2017), Mazucheli and Dey (2017) and references cited therein. To the best of our knowledge, such bias-corrected estimators have not yet been explored for the UG distribution in the literature.

In this paper, we propose two bias corrected MLEs for the two shape parameters of the UG distribution and illustrate their performance. First, we focus on the analytical methodology suggested by Cox and Snell (1968), which is called "corrective" approach and derive "bias adjusted" MLEs of second order where the bias-correction is obtained by subtracting the bias (estimated at the MLE of the parameters) from the true value of MLEs. The second one is based on Efron (1982) parametric Bootstrap resampling method which is also second-order bias correction. In this method bias correction is performed numerically without deriving analytical expression for the bias function. Readers can find another analytically bias-corrected MLEs in the literature which is based on "preventive" approach suggested by Firth (1993) where bias of the MLEs can be reduced to order $\mathcal{O}(n^{-2})$. However, this method requires modification of the score vector of the log-likelihood function before solving for the MLEs, and therefore, we have not attempted the approach in this paper. The effectiveness of these two bias correction, in terms of both bias reduction and its impact on root mean squared error, is compared with classical MLEs. It is apparent from the simulation study that the proposed estimators are quite accurate even for small sample sizes and are superior to classical MLEs in terms of their bias and root mean squared errors. Especially, they have simple mathematical expressions, which makes them attractive and easy to compute.

The remainder of this paper is organized as follows. In Sections 2 and 3, we summarize the maximum-likelihood estimation (MLEs) method and they bias-corrected estimators. A Monte Carlo simulation experiment that compares the Cox-Snell bias adjusted estimators with the Bootstrap bias-corrected estimators is discussed in Section 4. In Section 5, applications considering two real data sets are presented for illustrative purposes. Finally, Section 6 concludes the paper.

2. Maximum-likelihood estimation

In this section, we obtain the MLEs and the expected Fisher information matrix from complete samples for the UG distribution. Let $\boldsymbol{y} = (y_1, \ldots, y_n)$ be a random sample from (3), the log-likelihood function, apart constant terms, can be expressed as:

$$l(\boldsymbol{\Theta} \mid \boldsymbol{y}) \propto n\,\alpha \log \beta - n \log \Gamma(\alpha) + \beta \sum_{i=1}^{n} y_i + \alpha \sum_{i=1}^{n} \log(-\log y_i) \tag{4}$$

where $\boldsymbol{\Theta} = (\alpha, \beta)$. It is well known that the MLEs $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β, respectively can be obtained by the maximization of (1), or equivalently solving the following nonlinear equations:

$$\frac{\partial}{\partial \alpha} l(\boldsymbol{\Theta} \mid \boldsymbol{y}) = n \log \beta - n \psi(\alpha) + \sum_{i=1}^{n} \log(-\log y_i) \tag{5}$$

$$\frac{\partial}{\partial \beta} l(\boldsymbol{\Theta} \mid \boldsymbol{y}) = \frac{n \beta}{\alpha} + \sum_{i=1}^{n} \log y_i \tag{6}$$

where $\psi(\cdot)$ denotes the digamma function, defined as $\psi(u) = \frac{d}{du} \log \Gamma(u)$.

The standard statistical theory suggests that the MLEs $\widehat{\Theta}$ of Θ is asymptotically normally distributed with mean Θ and covariance matrix given by the inverse of the expected Fisher information matrix. The expected Fisher information matrix of unit-Gamma distribution is given by:

$$\boldsymbol{I}(\boldsymbol{\Theta} \mid \boldsymbol{y}) = n \begin{bmatrix} \psi'(\alpha) & \dfrac{1}{\beta} \\ \dfrac{1}{\beta} & \dfrac{\alpha}{\beta^2} \end{bmatrix} \tag{7}$$

and the corresponding inverse is:

$$\boldsymbol{I}^{-1}(\boldsymbol{\Theta} \mid \boldsymbol{y}) = \frac{1}{n} \begin{bmatrix} \dfrac{\alpha}{\psi'(\alpha)\,\alpha - 1} & \dfrac{\beta}{\psi'(\alpha)\,\alpha - 1} \\ \dfrac{\beta}{\psi'(\alpha)\,\alpha - 1} & \dfrac{\psi'(\alpha)\,\beta^2}{\psi'(\alpha)\,\alpha - 1} \end{bmatrix} \tag{8}$$

where $\psi'(\cdot)$ denotes the trigamma function, define as $\psi'(u) = \frac{d}{du} \psi(u)$.

From (7) we note that α and β are not orthogonal, which means that the MLEs $\widehat{\alpha}$ and $\widehat{\beta}$ are not asymptotically independent. It is also interesting to note that since $\boldsymbol{I}(\boldsymbol{\Theta} \mid \boldsymbol{y})$ is data independent then it is equal to the observed Fisher information matrix.

3. Bias-corrected MLEs

In this section, we shall derive closed-form expressions for the second order biases for the parameters of unit-Gamma distribution using the methodology proposed by Cox and Snell (1968). These authors demonstrated that when the sample data are independent, but not necessarily identically distributed, the bias of the s–th element of the MLE of $\boldsymbol{\Theta}$, $\widehat{\boldsymbol{\Theta}}$, can be expressed as:

$$\mathcal{B}(\widehat{\Theta}_s) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \kappa^{si} \kappa^{jl} \left[0.5 \kappa_{ijl} + \kappa_{ij,l}\right] + \mathcal{O}(n^{-2}), \tag{9}$$

where $s = 1, \ldots, p$, κ^{ij} is the (i, j)–th elements of the inverse of the expected Fisher information, $\kappa_{ijl} = \mathbb{E}[\frac{\partial^3}{\partial \boldsymbol{\Theta}_i \partial \boldsymbol{\Theta}_j \partial \boldsymbol{\Theta}_l} l(\boldsymbol{\Theta} \mid \boldsymbol{y})]$ and $\kappa_{ij,l} = \mathbb{E}[\frac{\partial^2}{\partial \boldsymbol{\Theta}_i \partial \boldsymbol{\Theta}_j} l(\boldsymbol{\Theta} \mid \boldsymbol{y})\, \frac{\partial}{\partial \boldsymbol{\Theta}_l} l(\boldsymbol{\Theta} \mid \boldsymbol{y})]$.

After extensive algebraic manipulations we obtain $\kappa_{111} = \frac{2 n \alpha}{\beta^3}$, $\kappa_{112} = \kappa_{121} = \kappa_{211} = -\frac{n}{\beta^2}$ and $\kappa_{222} = -n\, \psi''(\alpha)$ where $\psi''(\cdot)$ denotes the tetragamma function, define as $\psi''(u) = \frac{d}{du} \psi'(u)$. Since the Fisher information matrix is data independent, all others terms that will be used in (9) are equal to zero. Explicitly, the second-order bias of the MLE of α and β are written, respectively, as:

$$\mathcal{B}(\widehat{\alpha}) = \frac{0.5\, \alpha\, \psi'(\alpha) - 0.5\, \alpha^2\, \psi''(\alpha) - 1}{n[\alpha\, \psi'(\alpha) - 1]^2} \tag{10}$$

and:

$$\mathcal{B}(\widehat{\beta}) = \frac{\beta[\alpha(\psi'(\alpha))^2 - 1.5\,\psi'(\alpha) - 0.5\,\alpha\,\psi''(\alpha)]}{n[\alpha\,\psi'(\alpha) - 1]^2}. \tag{11}$$

Therefore, using (10) and (11) we define the bias-corrected estimators (BCE) of $\widehat{\alpha}$ and $\widehat{\beta}$ are, respectively $\widehat{\alpha}_{BCE} = \widehat{\alpha} - \widehat{\mathcal{B}}(\widehat{\alpha})$ and $\widehat{\beta}_{BCE} = \widehat{\beta} - \widehat{\mathcal{B}}(\widehat{\beta})$.

It should be point out that $\widehat{\alpha}_{BCE}$ and $\widehat{\beta}_{BCE}$ have bias of order $\mathcal{O}(n^{-2})$ so it is expected that they have superior sampling properties relative to $\widehat{\alpha}$ and $\widehat{\beta}$. Another alternative bias-corrected estimates we can consider the parametric Bootstrap methodology for bias reduction (PBE) which was introduced by Efron (1982). In the case of an arbitrary parameter Θ the estimated bias of $\widehat{\Theta}$ is defined as:

$$\widehat{\mathcal{B}}(\widehat{\Theta}) = \frac{1}{B}\sum_{j=1}^{B} \Theta_{(j)} - \widehat{\Theta} \tag{12}$$

where $\widehat{\Theta}_{(j)}$ is the MLE of Θ obtained from the j-th Bootstrap sample, generated from (3) and using the MLE $\widehat{\Theta}$ as the true value. Thus, the Bootstrap bias-corrected estimator is:

$$\widehat{\Theta}_{PBE} = 2\,\widehat{\Theta} - \frac{1}{B}\sum_{j=1}^{B} \widehat{\Theta}_{(j)}. \tag{13}$$

It is noteworthy that the PBE does not involve analytical derivatives, since the bias estimation is obtained numerically. However, this approach also provides a second order bias-correction. A common feature of the two methods is that they are corrective, rather than preventive as the method proposed by Firth (1993). Interested readers may refer to Ferrari and Cribari-Neto (1998) for a discussion of the second order correctness of the Bootstrap bias-correction and its relation to the Cox-Snell methodology.

4. Simulation study

In this section, we carry out Monte Carlo simulations to compare the finite-sample behavior of the MLEs and their bias-corrected versions obtained by the Cox-Snell methodology (BCE) and parametric Bootstrap scheme (PBE) for the parameters of UG distribution. The comparison is based on the estimated bias and the estimated root mean-squared error criteria. The Monte Carlo experiments were performed by taking samples sizes $n = 10, 20, 30, 40$ and 50, $\alpha = 0.5, 1.0, 1.5$ and 2.0 and $\beta = 0.5, 1.0, 2.0, 3.0$ and 5.0. For each combination of n, α and β, we generated X as Gamma distribution and apply transformation $Y = e^{-X}$. The number of Monte Carlo replications is fixed at $M = 10.000$ and $B = 1000$ Bootstrap replicates.

All simulations were conducted in Ox Console (Doornik 2007), using the MaxBFGS function to obtain the MLEs of α and β. The results from the simulations (estimated biases and root mean-squared errors) are reported in Tables 1–4.

From Table 1, we observe that the MLEs of β are highly biased, while for α, bias is moderate, particularly when the sample size is small. For example, when $n = 10$, $\alpha = 0.5$, the biases of the MLEs of α is approximately 15%. Similar results are observed in Tables 2–4. Similarly, when $n = 10$, $\beta = 0.5$, the biases of the MLEs of β is approximately 31%. Thus the estimators $\widehat{\alpha}_{BCE}$, $\widehat{\beta}_{BCE}$ and $\widehat{\alpha}_{PBE}$, $\widehat{\beta}_{BCE}$ clearly outperform the MLEs as far as bias goes. These estimators achieve substantial bias reduction, especially for the small and moderate sample sizes and therefore we consider them as better alternatives of the MLEs for α and β. Similar results for β are observed. We also observe that the bias-corrected estimates are closer to the true

Table 1. Estimated bias (root mean-squared error) for β and α, ($\alpha = 0.5$).

		Estimates of β			Estimates of α		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.305 (0.732)	−0.006 (0.448)	−0.153 (0.391)	0.151 (0.371)	−0.000 (0.243)	−0.058 (0.209)
	20	0.120 (0.316)	−0.005 (0.242)	−0.030 (0.236)	0.060 (0.178)	−0.002 (0.145)	−0.013 (0.141)
	30	0.078 (0.227)	−0.001 (0.189)	−0.011 (0.186)	0.040 (0.132)	0.000 (0.115)	−0.004 (0.114)
	40	0.056 (0.181)	−0.001 (0.157)	−0.006 (0.156)	0.029 (0.107)	−0.000 (0.097)	−0.003 (0.096)
	50	0.045 (0.155)	0.000 (0.138)	−0.003 (0.137)	0.023 (0.094)	0.000 (0.086)	−0.001 (0.085)
1.0	10	0.614 (1.400)	−0.010 (0.842)	−0.303 (0.744)	0.152 (0.371)	0.001 (0.243)	−0.057 (0.210)
	20	0.255 (0.649)	0.003 (0.495)	−0.048 (0.480)	0.065 (0.182)	0.002 (0.147)	−0.009 (0.143)
	30	0.158 (0.456)	0.001 (0.379)	−0.019 (0.374)	0.041 (0.132)	0.001 (0.114)	−0.003 (0.113)
	40	0.113 (0.367)	−0.001 (0.319)	−0.012 (0.317)	0.030 (0.108)	0.000 (0.097)	−0.002 (0.096)
	50	0.087 (0.312)	−0.003 (0.279)	−0.009 (0.278)	0.023 (0.093)	0.000 (0.085)	−0.001 (0.085)
2.0	10	1.258 (3.002)	0.001 (1.826)	−0.594 (1.587)	0.150 (0.371)	−0.001 (0.244)	−0.059 (0.210)
	20	0.506 (1.306)	0.003 (0.998)	−0.098 (0.970)	0.062 (0.176)	−0.001 (0.143)	−0.011 (0.140)
	30	0.320 (0.933)	0.006 (0.775)	−0.035 (0.765)	0.039 (0.131)	−0.001 (0.114)	−0.005 (0.113)
	40	0.240 (0.747)	0.011 (0.646)	−0.010 (0.641)	0.030 (0.107)	0.000 (0.096)	−0.002 (0.096)
	50	0.190 (0.635)	0.010 (0.563)	−0.003 (0.561)	0.023 (0.093)	−0.000 (0.085)	−0.001 (0.085)
3.0	10	1.901 (4.459)	0.012 (2.703)	−0.881 (2.329)	0.152 (0.380)	0.000 (0.250)	−0.057 (0.215)
	20	0.773 (1.955)	0.016 (1.488)	−0.137 (1.443)	0.064 (0.180)	0.001 (0.145)	−0.010 (0.142)
	30	0.479 (1.366)	0.008 (1.131)	−0.052 (1.117)	0.040 (0.131)	0.001 (0.114)	−0.004 (0.113)
	40	0.342 (1.085)	0.000 (0.940)	−0.032 (0.935)	0.030 (0.108)	0.000 (0.097)	−0.002 (0.096)
	50	0.276 (0.929)	0.007 (0.825)	−0.012 (0.822)	0.024 (0.093)	0.001 (0.085)	−0.001 (0.085)
5.0	10	3.062 (7.372)	−0.052 (4.503)	−1.506 (3.933)	0.146 (0.374)	−0.004 (0.247)	−0.060 (0.214)
	20	1.212 (3.211)	−0.037 (2.466)	−0.291 (2.401)	0.061 (0.179)	−0.002 (0.145)	−0.012 (0.142)
	30	0.728 (2.201)	−0.050 (1.837)	−0.149 (1.819)	0.036 (0.128)	−0.003 (0.112)	−0.007 (0.111)
	40	0.527 (1.791)	−0.039 (1.563)	−0.091 (1.554)	0.027 (0.106)	−0.002 (0.096)	−0.004 (0.095)
	50	0.409 (1.539)	−0.036 (1.380)	−0.069 (1.377)	0.021 (0.092)	−0.002 (0.085)	−0.003 (0.085)

Table 2. Estimated bias (root mean-squared error) for β and α, ($\alpha = 1.0$).

		Estimates of β			Estimates of α		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.248 (0.583)	−0.002 (0.367)	−0.112 (0.319)	0.343 (0.844)	−0.005 (0.544)	−0.153 (0.474)
	20	0.102 (0.266)	−0.001 (0.208)	−0.019 (0.202)	0.141 (0.398)	−0.003 (0.318)	−0.028 (0.310)
	30	0.062 (0.188)	−0.002 (0.159)	−0.009 (0.158)	0.086 (0.284)	−0.005 (0.245)	−0.015 (0.243)
	40	0.044 (0.152)	−0.003 (0.134)	−0.007 (0.133)	0.061 (0.231)	−0.005 (0.207)	−0.011 (0.206)
	50	0.036 (0.132)	−0.001 (0.120)	−0.004 (0.119)	0.049 (0.203)	−0.003 (0.185)	−0.006 (0.185)
1.0	10	0.488 (1.118)	−0.008 (0.700)	−0.226 (0.616)	0.347 (0.837)	−0.002 (0.537)	−0.150 (0.466)
	20	0.198 (0.519)	−0.005 (0.406)	−0.043 (0.396)	0.140 (0.388)	−0.004 (0.309)	−0.029 (0.302)
	30	0.128 (0.378)	−0.001 (0.319)	−0.016 (0.315)	0.091 (0.288)	0.000 (0.247)	−0.010 (0.244)
	40	0.092 (0.303)	−0.002 (0.267)	−0.010 (0.265)	0.066 (0.233)	−0.000 (0.207)	−0.005 (0.206)
	50	0.071 (0.259)	−0.002 (0.233)	−0.007 (0.232)	0.052 (0.199)	−0.000 (0.181)	−0.004 (0.181)
2.0	10	0.995 (2.341)	−0.003 (1.474)	−0.442 (1.276)	0.351 (0.859)	0.001 (0.553)	−0.148 (0.475)
	20	0.406 (1.049)	−0.003 (0.819)	−0.077 (0.798)	0.144 (0.399)	−0.000 (0.318)	−0.026 (0.310)
	30	0.251 (0.739)	−0.006 (0.623)	−0.035 (0.617)	0.089 (0.287)	−0.002 (0.246)	−0.012 (0.244)
	40	0.192 (0.617)	0.004 (0.541)	−0.012 (0.538)	0.067 (0.237)	0.001 (0.211)	−0.005 (0.210)
	50	0.152 (0.537)	0.004 (0.483)	−0.006 (0.482)	0.052 (0.204)	−0.001 (0.186)	−0.004 (0.185)
3.0	10	1.486 (3.383)	−0.009 (2.113)	−0.667 (1.853)	0.348 (0.823)	−0.001 (0.526)	−0.149 (0.456)
	20	0.613 (1.599)	−0.000 (1.250)	−0.112 (1.217)	0.147 (0.402)	0.002 (0.319)	−0.023 (0.311)
	30	0.391 (1.149)	0.005 (0.970)	−0.039 (0.959)	0.095 (0.291)	0.003 (0.249)	−0.007 (0.246)
	40	0.286 (0.925)	0.005 (0.812)	−0.020 (0.807)	0.069 (0.235)	0.003 (0.208)	−0.003 (0.207)
	50	0.232 (0.808)	0.010 (0.726)	−0.005 (0.723)	0.056 (0.204)	0.003 (0.185)	−0.000 (0.185)
5.0	10	2.428 (5.596)	−0.051 (3.505)	−1.139 (3.092)	0.335 (0.806)	−0.011 (0.517)	−0.157 (0.454)
	20	0.998 (2.688)	−0.022 (2.113)	−0.208 (2.059)	0.139 (0.399)	−0.005 (0.320)	−0.030 (0.312)
	30	0.616 (1.893)	−0.025 (1.606)	−0.098 (1.589)	0.086 (0.287)	−0.005 (0.248)	−0.015 (0.245)
	40	0.442 (1.523)	−0.025 (1.346)	−0.064 (1.338)	0.061 (0.234)	−0.005 (0.209)	−0.011 (0.208)
	50	0.329 (1.291)	−0.037 (1.172)	−0.061 (1.168)	0.046 (0.201)	−0.006 (0.184)	−0.010 (0.184)

Table 3. Estimated bias (root mean-squared error) for β and α, ($\alpha = 1.5$).

		Estimates of β			Estimates of α		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.234 (0.545)	− 0.001 (0.344)	− 0.102 (0.299)	0.554 (1.326)	− 0.003 (0.846)	− 0.241 (0.733)
	20	0.097 (0.257)	0.000 (0.202)	− 0.017 (0.196)	0.229 (0.634)	− 0.001 (0.503)	− 0.041 (0.490)
	30	0.061 (0.184)	0.000 (0.156)	− 0.007 (0.155)	0.144 (0.457)	− 0.001 (0.391)	− 0.017 (0.387)
	40	0.043 (0.149)	− 0.001 (0.132)	− 0.005 (0.131)	0.101 (0.370)	− 0.004 (0.329)	− 0.013 (0.328)
	50	0.034 (0.129)	− 0.001 (0.117)	− 0.003 (0.116)	0.080 (0.319)	− 0.003 (0.290)	− 0.009 (0.290)
1.0	10	0.479 (1.083)	0.006 (0.678)	− 0.198 (0.591)	0.566 (1.324)	0.005 (0.841)	− 0.234 (0.731)
	20	0.201 (0.517)	0.006 (0.405)	− 0.029 (0.393)	0.233 (0.631)	0.003 (0.499)	− 0.038 (0.486)
	30	0.126 (0.369)	0.004 (0.312)	− 0.010 (0.309)	0.148 (0.455)	0.003 (0.388)	− 0.013 (0.384)
	40	0.091 (0.298)	0.002 (0.262)	− 0.005 (0.261)	0.107 (0.369)	0.001 (0.327)	− 0.007 (0.325)
	50	0.073 (0.255)	0.003 (0.229)	− 0.002 (0.229)	0.085 (0.316)	0.002 (0.287)	− 0.004 (0.285)
2.0	10	0.922 (2.120)	− 0.012 (1.334)	− 0.417 (1.172)	0.550 (1.286)	− 0.006 (0.817)	− 0.243 (0.715)
	20	0.400 (1.034)	0.012 (0.809)	− 0.057 (0.787)	0.242 (0.640)	0.010 (0.505)	− 0.031 (0.491)
	30	0.258 (0.740)	0.013 (0.624)	− 0.015 (0.616)	0.156 (0.464)	0.010 (0.394)	− 0.006 (0.389)
	40	0.188 (0.600)	0.010 (0.526)	− 0.005 (0.524)	0.114 (0.374)	0.007 (0.330)	− 0.001 (0.328)
	50	0.146 (0.512)	0.006 (0.461)	− 0.003 (0.459)	0.087 (0.321)	0.004 (0.291)	− 0.002 (0.290)
3.0	10	1.393 (3.195)	− 0.012 (2.008)	− 0.621 (1.756)	0.552 (1.295)	− 0.004 (0.823)	− 0.242 (0.718)
	20	0.585 (1.492)	0.005 (1.165)	− 0.099 (1.134)	0.233 (0.620)	0.003 (0.490)	− 0.038 (0.476)
	30	0.377 (1.078)	0.011 (0.908)	− 0.030 (0.898)	0.148 (0.444)	0.003 (0.377)	− 0.014 (0.373)
	40	0.287 (0.887)	0.020 (0.776)	− 0.003 (0.771)	0.112 (0.367)	0.006 (0.324)	− 0.003 (0.322)
	50	0.231 (0.772)	0.020 (0.692)	0.006 (0.689)	0.089 (0.319)	0.006 (0.288)	0.000 (0.287)
5.0	10	2.365 (5.325)	0.010 (3.332)	− 1.007 (2.930)	0.557 (1.293)	− 0.001 (0.820)	− 0.238 (0.718)
	20	0.987 (2.514)	0.018 (1.963)	− 0.157 (1.909)	0.234 (0.620)	0.003 (0.490)	− 0.038 (0.476)
	30	0.617 (1.798)	0.008 (1.518)	− 0.061 (1.500)	0.147 (0.444)	0.001 (0.378)	− 0.015 (0.374)
	40	0.449 (1.464)	0.005 (1.288)	− 0.032 (1.282)	0.107 (0.363)	0.001 (0.321)	− 0.008 (0.319)
	50	0.351 (1.252)	0.002 (1.129)	− 0.022 (1.125)	0.083 (0.311)	− 0.000 (0.282)	− 0.006 (0.281)

Table 4. Estimated bias (root mean-squared error) for β and α, ($\alpha = 2.0$).

		Estimates of β			Estimates of α		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.233 (0.533)	0.003 (0.335)	− 0.096 (0.292)	0.782 (1.839)	0.009 (1.167)	− 0.322 (1.013)
	20	0.096 (0.252)	0.001 (0.198)	− 0.015 (0.193)	0.322 (0.874)	0.004 (0.691)	− 0.052 (0.673)
	30	0.061 (0.181)	0.002 (0.153)	− 0.005 (0.151)	0.207 (0.629)	0.006 (0.535)	− 0.016 (0.529)
	40	0.043 (0.146)	0.000 (0.129)	− 0.003 (0.128)	0.146 (0.509)	0.000 (0.451)	− 0.011 (0.449)
	50	0.035 (0.126)	0.001 (0.114)	− 0.001 (0.114)	0.116 (0.441)	0.002 (0.400)	− 0.006 (0.398)
1.0	10	0.452 (1.019)	− 0.004 (0.639)	− 0.200 (0.560)	0.762 (1.734)	− 0.006 (1.092)	− 0.333 (0.958)
	20	0.193 (0.499)	0.004 (0.391)	− 0.029 (0.380)	0.325 (0.858)	0.007 (0.675)	− 0.049 (0.657)
	30	0.124 (0.362)	0.005 (0.306)	− 0.008 (0.303)	0.209 (0.623)	0.009 (0.529)	− 0.014 (0.523)
	40	0.090 (0.291)	0.004 (0.256)	− 0.003 (0.254)	0.154 (0.503)	0.007 (0.444)	− 0.004 (0.441)
	50	0.070 (0.250)	0.002 (0.226)	− 0.003 (0.225)	0.119 (0.436)	0.004 (0.394)	− 0.003 (0.393)
2.0	10	0.874 (2.134)	− 0.029 (1.361)	− 0.415 (1.197)	0.735 (1.820)	− 0.024 (1.168)	− 0.347 (1.021)
	20	0.370 (0.983)	− 0.005 (0.773)	− 0.072 (0.753)	0.314 (0.860)	− 0.003 (0.682)	− 0.058 (0.664)
	30	0.236 (0.696)	− 0.001 (0.589)	− 0.028 (0.583)	0.199 (0.604)	− 0.001 (0.514)	− 0.023 (0.508)
	40	0.173 (0.574)	0.000 (0.506)	− 0.014 (0.503)	0.148 (0.502)	0.002 (0.444)	− 0.010 (0.441)
	50	0.137 (0.491)	0.000 (0.443)	− 0.009 (0.441)	0.117 (0.430)	0.002 (0.390)	− 0.006 (0.388)
3.0	10	1.350 (3.093)	− 0.016 (1.946)	− 0.606 (1.708)	0.761 (1.778)	− 0.006 (1.127)	− 0.335 (0.985)
	20	0.562 (1.472)	− 0.002 (1.156)	− 0.102 (1.124)	0.321 (0.860)	0.004 (0.679)	− 0.052 (0.659)
	30	0.360 (1.067)	0.004 (0.903)	− 0.036 (0.894)	0.208 (0.621)	0.007 (0.528)	− 0.015 (0.522)
	40	0.260 (0.862)	0.001 (0.760)	− 0.021 (0.756)	0.152 (0.505)	0.006 (0.445)	− 0.006 (0.443)
	50	0.202 (0.740)	− 0.002 (0.669)	− 0.015 (0.668)	0.119 (0.434)	0.004 (0.392)	− 0.003 (0.392)
5.0	10	2.325 (5.362)	0.025 (3.378)	− 0.968 (2.923)	0.768 (1.777)	− 0.001 (1.124)	− 0.330 (0.979)
	20	0.920 (2.401)	− 0.018 (1.884)	− 0.185 (1.836)	0.310 (0.834)	− 0.006 (0.659)	− 0.062 (0.643)
	30	0.590 (1.741)	− 0.002 (1.473)	− 0.069 (1.458)	0.202 (0.605)	0.002 (0.514)	− 0.021 (0.509)
	40	0.426 (1.415)	− 0.006 (1.247)	− 0.042 (1.240)	0.146 (0.494)	0.001 (0.437)	− 0.011 (0.434)
	50	0.337 (1.217)	− 0.003 (1.099)	− 0.026 (1.096)	0.116 (0.428)	0.002 (0.387)	− 0.006 (0.386)

Table 5. Integrated bias squared norm.

	Estimates of β			Estimates of α		
n	MLE	BCE	PBE	MLE	BCE	PBE
10	1.4379	0.0198	0.6581	0.5076	0.0070	0.2208
20	0.5857	0.0126	0.1161	0.2125	0.0040	0.0363
30	0.3654	0.0137	0.0506	0.1356	0.0043	0.0130
40	0.2657	0.0120	0.0299	0.0988	0.0037	0.0072
50	0.2078	0.0132	0.0228	0.0777	0.0029	0.0046

parameter values than the unadjusted estimates as sample size increases. Additionally, the root-mean squared errors of the corrected estimates are smaller than those of the uncorrected estimates. Thus, it is clear that the estimators BCE and PBE also achieve mean-squared error reduction. Note that the root mean-squared errors decrease with n, as expected.

Finally, we assess the overall performance of each estimators with respect to the bias and root mean-squared error through two measures proposed by Cribari-Neto and Vasconcellos (2002). The authors called these measures as integrated bias squared and average root mean-squared error. They are defined as follows:

$$IBSQ_{(k)} = \sqrt{\frac{1}{20}\sum_{h=1}^{20}(r_{h,k})^2} \quad \text{and} \quad ARMSE_{(k)} = \frac{1}{20}\sum_{h=1}^{20} RMSE_{h,k}$$

where $r_{h,k}$ and $RMSE_{h,k}$ are the biases and the root mean-squared errors. The numerical results of these quantities are presented in Tables 5 and 6.

We also observe from Tables 5 and 6 that integrated bias squared and average root mean-squared error of the corrected estimates (BCE and PBE) are smaller than MLEs for both the parameters α and β. Thus, these simulation results show that second-order bias reduction can be quite successful in bringing the corrected estimates closer to their true values.

5. Applications

In this section, we shall analyze two real data applications in order to illustrate the bias-corrected estimators proposed for the parameters of unit-Gamma distribution. For the sake of comparison, we also fitted the Beta and Kumaraswamy distribution and their corresponding PDFs are given by

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} \quad \text{and} \quad f(x \mid \alpha, \beta) = \alpha\,\beta\, x^{\alpha-1}\,(1-x^{\alpha})^{\beta-1}$$

respectively, where $\alpha > 0$ and $\beta > 0$ are shape parameters.

Table 6. Average root mean-squared error.

	Estimates of β			Estimates of α		
n	MLE	BCE	PBE	MLE	BCE	PBE
10	3.3488	2.0789	1.8166	1.1981	0.7624	0.6646
20	1.5288	1.1884	1.1566	0.5745	0.4548	0.4429
30	1.0814	0.9099	0.9000	0.4138	0.3525	0.3486
40	0.8765	0.7687	0.7642	0.3371	0.2986	0.2969
50	0.7516	0.6764	0.6742	0.2908	0.2638	0.2629

Table 7. MLEs and bias-corrected MLEs (standard-error) — flood level data.

Estimators	α	β
MLE	8.7312 (2.7099)	9.7255 (3.1070)
BCE	7.4542 (2.3064)	8.2474 (2.6397)
PBE	7.4420 (2.3025)	8.1908 (2.6216)

The first data set is from Dumonceaux and Antle (1973) and refer to 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania.

The point estimates of α and β along with standard errors obtained by all the considered methods are summarized in Table 7 for the data set 1. We can note from the results of Table 7 that the BCE and PBE estimates of α and β are smaller than the uncorrected MLEs, which suggest that the estimation by maximum-likelihood method over estimates α and β. Furthermore, we also observe that the corrected MLEs have smaller standard errors for the parameters. This means that the proposed estimators performed more efficiently than the MLE. It is worth pointing out that all the estimations are obviously different, which indicates that even when the sample size is small or moderate, the bias correction is still necessary because it contains useful information.

The estimated parameters and their standard errors (in parentheses) for the Beta distribution are $\widehat{\alpha} = 6.7564$ (2.0944) and $\widehat{\beta} = 9.1110$ (2.8515), whilst for the Kumaraswamy distribution are $\widehat{\alpha} = 3.3631$ (0.6033) and $\widehat{\beta} = 11.7888$ (5.3595).

The choice of the most appropriate distribution is based on the values of the likelihood-based statistics such as Akaike's Information Criterion (AIC), corrected Akaike's Information Criterion (AIC_c), consistent Akaike's Information Criterion ($CAIC$) and Hannan-Quinn Information Criterion ($HQIC$) and the goodness-of-fit measures like Kolmogorov-Smirnov statistic (KS), Anderson-Darling statistic (AD) and Cramér-von Mises statistic (CvM) evaluated at analytical bias-corrected MLEs for the unit-Gamma distribution. It is noteworthy that all the above mentioned criteria favors the model in respect of the lowest values.

From Table 8 we can note that the unit-Gamma and Beta distribution shows quite similar values to statistics considered, whereas the Kumaraswamy distribution shows the worst fit.

The second data set corresponds to twelve core samples from petroleum reservoirs that were sampled by four cross-sections, and there are 48 observations. Each core sample was measured for permeability and each cross-section has the following variables: the total area of pores, the total perimeter of pores and shape. We shall analyze the shape perimeter by squared (area) variable. It should be noted that this data can be found in R Core Team (2017), especially on `data.frame` named `rock`.

Table 8. Likelihood-based statistics and goodness-of-fit measures (p-values) — flood level data.

	Distribution		
Statistics	unit-Gamma	Beta	Kumaraswamy
AIC	− 24.1232	− 24.1245	− 21.7324
AIC_c	− 23.4173	− 23.4186	− 21.0265
$CAIC$	− 21.1317	− 21.1330	− 18.7409
$HQIC$	− 23.7344	− 23.7357	− 21.3436
KS	0.1958 (0.4269)	0.1988 (0.4082)	0.2109 (0.3360)
CvM	0.1276 (0.4689)	0.1236 (0.4847)	0.1636 (0.3529)
AD	0.7202 (0.5402)	0.7327 (0.5302)	0.9321 (0.3936)

Table 9. MLEs and bias-corrected MLEs (standard-error) — petroleum reservoirs data.

Estimators	α	β
MLE	17.9499 (3.6305)	11.3088 (2.3195)
BCE	16.8418 (3.4043)	10.5975 (2.1743)
PBE	16.9396 (3.4243)	10.6460 (2.1842)

Table 10. Likelihood-based statistics and goodness-of-fit measures (p-values) — petroleum reservoirs data.

	Distribution		
Statistics	unit-Gamma	Beta	Kumaraswamy
AIC	− 108.1190	− 107.2004	− 100.9821
AIC_c	− 107.8523	− 106.9338	− 100.7154
CAIC	− 103.3766	− 102.4580	− 96.2397
HQIC	− 106.7047	− 105.7862	− 99.5678
KS	0.1354 (0.3421)	0.1428 (0.2820)	0.1532 (0.2101)
CvM	0.1152 (0.5169)	0.1301 (0.4578)	0.2069 (0.2549)
AD	0.6879 (0.5682)	0.7770 (0.4972)	1.2931 (0.2345)

Table 9 lists the MLEs and the bias-corrected MLEs for α and β along with standard errors obtained by all the considered methods. It is observed that BCE estimators provide the lowest standard errors. We also note that the BCE and PBE estimates of α and β are smaller than the uncorrected MLEs. Thus it is clear that the MLEs overestimates α and β. For the Beta distribution the ML estimates are $\widehat{\alpha} = 5.9417\ (1.1813)$ and $\widehat{\beta} = 21.2056\ (4.3468)$, whereas for Kumaraswamy distribution the MLEs are $\widehat{\alpha} = 2.7108\ (0.2911)$ and $\widehat{\beta} = 44.1196\ (17.2198)$. Table 10 shows the statistics used for discrimination among the distributions. We note that the UG distribution has the lowest values of these statistics and thereby provide the best fit.

6. Conclusions

In this paper, we have adopted a "corrective" approach to derive closed-form expressions for the second order biases of the MLEs of the parameters of the unit-Gamma distribution. In addition, we have also considered an alternative bias-correction mechanism through Efron's Bootstrap resampling. The numerical evidence shows the proposed bias-corrected procedures are very effective. The MLEs of both the parameters are positively biased in small and moderate samples. Additionally, the proposed estimators (both BCE and PBE) are quite attractive because they outperform MLEs in terms of biases, integrated bias squared norm and root mean-squared error. Results also reveal that the integrated bias squared and root mean-squared errors of all the estimators decrease as the sample size n increases. Furthermore, we found that Bootstrap bias correction scheme is less effective than the analytic correction in terms of bias reduction. Thus, the analytic bias correction is recommended over the Bootstrap alternative, except in extreme cases when the sample size is very small.

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