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The unit-Birnbaum-Saunders distribution with applications

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Abstract

In this paper a new probability distribution supported on unit interval is introduced. This distribution arises from the Birnbaum and Saunders distribution. It depends on two parameters and can be considered as an alternative to the classical Beta distribution and Kumaraswamy distribution. It presents the advantage of not including any additional parameter(s) or special function in its formulation and it has closed-form for the moments. The new transformed model, called the unit-Birnbaum-Saunders (UBS) distribution exhibits decreasing, upside down bathtub and then bathtub shaped density while the hazard rate function can be increaseing or bathtub shaped. The method of maximum likelihood and moments are used to estimate the model parameters. Monte Carlo simulation is carried out to examine the bias and root mean squared error of the maximum likelihood and moment estimators of the parameters. Finally, the potentiality of the model is studied using two real data sets.

Keywords: Birnbaum and Saunders distribution · Maximum likelihood estimators · Moment estimators · Monte Carlo simulation.

1. Introduction

Birnbaum and Saunders (1969a,b) developed an important two-parameter lifetime distribution for fatigue failure caused under cyclic stress. This distribution can be obtained using the Central Limit Theorem (CLT) for independently identically distributed random variables with finite second moments. Henceforth, we call this distribution as the BS distribution. The probability density function (p.d.f) of the BS distribution is unimodal possessing many attractive properties. Also, its close relationship to the normal distribution renders it as an alternative to the normal model for data with nonnegative support and positive skewness. That is why BS distribution finds wide applicability in the literature including lifetime, survival and environmental data analysis. For instance, Desmond (1985) considered the BS distribution as a biological model, Leiva et al. (2007) and Barros et al. (2008) presented an application in the medical field, and Podlaski (2008), Leiva et al. (2008, 2009, 2010), and Vilca et al. (2010) used this model for describing data from the forestry and environmental sciences. For more details about various developments on the BS distribution, one may refer to Johnson and Kotz (1995), Sanhueza (2008), Ng et al. (2006, 2003), Kundu et al. (2008), Balakrishnan et al. (2011) and the references cited therein.

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Many generalizations of the Birnbaum-Saunders distribution have been attempted by researchers. Notable among them are Patriota (2012) introduced scale mixture of Birnbaum-Saunders distribution using generalized CLT, Cordeiro and Lemonte (2011) introduced the $\beta-$Birnbaum-Saunders distribution, Leiva et al. (2009) proposed length biased version of Birnbaum Saunders distribution, Rieck and Nedelman (1991) introduced log Birnbaum-Saunders distribution and showed that it can be obtained as a special case of the sinh-normal distribution, Lemonte (2012) proposed log-Birnbaum-Saunders regression model with asymmetric errors, Lemonte (2013) studied multivariate Birnbaum Saunders regression model. Most of these models involve special functions or additional parameters in their formulation.

In order to overcome this problem, a new two-parameter distribution with bounded domain that can be considered as an alternative to the classical Beta and Kumaraswamy distributions is proposed here. We call this new model as unit-Birnbaum-Saunders distribution (UBS). It presents the advantage of not including any special function or additional parameters in its formulation. Other motivations are that it is capable of modeling increasing and bathtub shaped hazard rate and two real data applications show that it compares well with other two competing lifetime distributions in modeling environmental and milk production data.

The article is organized as follows. In Section 2 we introduce the UBS distribution and discuss some basic properties of this family of distributions. In Section 3 maximum likelihood and moment estimators of the unknown parameters along with asymptotic confidence intervals are obtained. Monte Carlo simulation results are presented in Section 4. The analysis of two real data sets have been presented in Section 5. Concluding remarks are finally made in Section 6.

2. Model Description

In this section, we describe a new bounded distribution, which arises from the Birnbaum-Saunders distribution by the logarithmic transformation of the type $X = \mathrm{e}^{-Y}$, where Y has the Birnbaum-Saunders distribution. This transformation is also considered in Grassia (1977) for the unit-Gamma distribution and Gómez-Déniz et al. (2013) for the log-Lindley distribution. Let Y be a non negative random variable with Birnbaum-Saunders distribution, then its probability density function is given by:

$$f(y) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{y}\right)^{1/2} + \left(\frac{\beta}{y}\right)^{3/2}\right]\exp\left[-\frac{1}{2\,\alpha^2}\left(\frac{y}{\beta}+\frac{\beta}{y}-2\right)\right] \tag{1}$$

where $y > 0$, $\beta > 0$ and $\alpha > 0$ are scale and shape parameters, respectively. By considering the transformation $X = \mathrm{e}^{-Y}$, we obtain a new distribution with support on $(0, 1)$, which we refer to as unit-BS distribution. Its probability density function is given by:

$$f(x) = \frac{1}{2\,x\,\alpha\,\beta\,\sqrt{2\,\pi}}\left[\left(-\frac{\beta}{\log x}\right)^{1/2} + \left(-\frac{\beta}{\log x}\right)^{3/2}\right]\exp\left[\frac{1}{2\alpha^2}\left(\frac{\log x}{\beta}+\frac{\beta}{\log x}+2\right)\right]. \tag{2}$$

The corresponding cumulative distribution function is given by:

$$F(x) = \Phi\left[\frac{1}{\alpha}\left\{\left(-\frac{\log x}{\beta}\right)^{1/2} - \left(-\frac{\beta}{\log x}\right)^{1/2}\right\}\right]. \tag{3}$$

The new distribution is flexible to model positively skewed data sets which can exhibits increasing or bathtub shaped hazard rate. Figure 1 displays some plots of the p.d.f and hazard rate function (h.r.f.) of the UBS for selected values of α and β. The plots reveal that the UBS density is decreasing, upside down bathtub and then bathtub shaped. The plots also indicate that the h.r.f. of UBS is increasing and bathtub shaped. One of the advantages of the UBS distribution over the BS distribution is that the latter cannot model phenomenon showing increasing and bathtub shape failure rate.

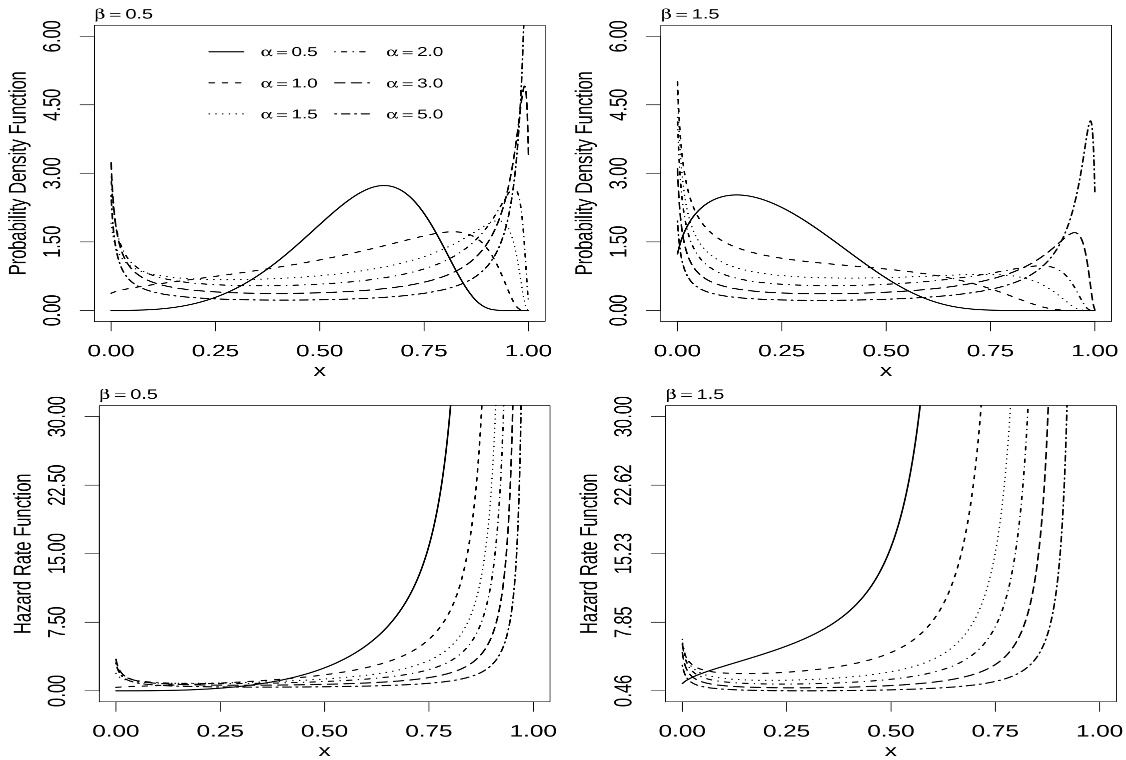

Figure 1. Probability density function and hazard rate function of UBS distribution considering different values of α and β.

We hardly need to emphasize the necessity and importance of the moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). If the random variable X has the UBS distribution, then the r-th moment about zero is given by:

$$\mathrm{E}\left(X^r\right) = \frac{2\,r\alpha^2\beta + \sqrt{2\,r\alpha^2\beta+1}+1}{4\,r\alpha^2\beta+1}\exp\left[-\frac{\sqrt{2\,r\alpha^2\beta+1}-1}{\alpha^2}\right]. \tag{4}$$

In particular, in Figure 2 we present the behavior of the mean, variance, skewness and kurtosis, as a function of β. The plots show that the skewness can be negative which is useful in modeling left skewed data.

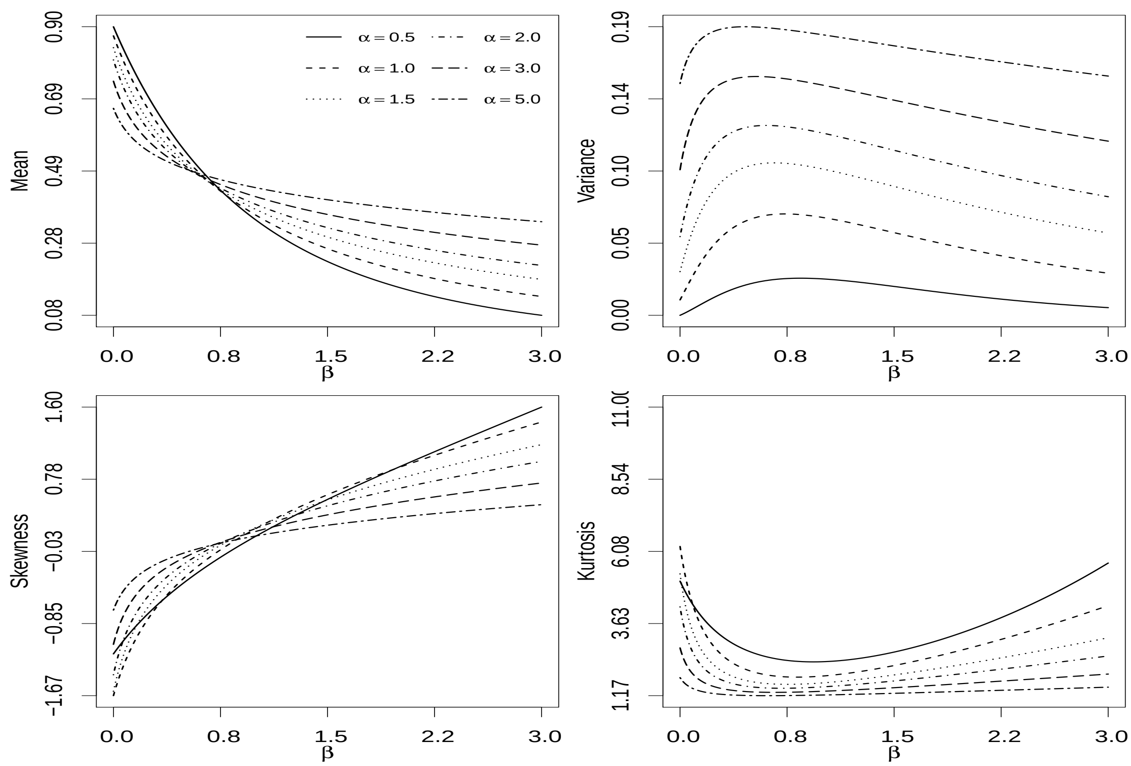

Figure 2. Behavior of the mean, variance, skewness and kurtosis of UBS distribution as a function of β.

It is important to emphasize that several methods can be used for deriving distributions with the support of unit-interval. For example, when the distribution of a random variable Y with support $[0, \infty)$, the distribution of $Y/(1+Y)$ has the support of unit-interval. In fact, by this procedure, the Beta distribution can be obtained from the F distribution, (Marshall and Olkin, 2007, see page 482). Again, if Y follows the generalized exponential distribution, (Gupta and Kundu, 1999), by using the transformation $\exp(-Y)$, we get the Kumaraswamy distribution, (Kumaraswamy, 1980).

3. Estimation

In this section, we consider the estimation of parameters α and β of the UBS distribution by the methods of moments and maximum likelihood and provide expressions for the associated Fisher information matrix.

3.1 Maximum Likelihood

Among the statistical inference methods, the maximum likelihood method is widely used due its desirable properties including consistency, asymptotic efficiency and invariance (Lehmann and Casella, 1998; Edwards, 1992). Under the maximum likelihood paradigm, the estimators are obtained by maximizing the log-likelihood function. Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ denote a random sample of size n from $X \sim \text{UBS}(\alpha, \beta)$. Then, the log-likelihood function for $\boldsymbol{\Theta} = (\alpha, \beta)^\top$, without constant terms, can be written as:

$$\ell(\boldsymbol{\Theta} \mid \mathbf{x}) \propto -n \log(\alpha\beta) + \sum_{i=1}^{n} \left[\left(\frac{\beta}{\log x_i} \right)^{1/2} + \left(\frac{\beta}{\log x_i} \right)^{3/2} \right] - \frac{1}{2\alpha^2} \sum_{i=1}^{n} \left(\frac{\log x_i}{\beta} + \frac{\beta}{\log x_i} - 2 \right).$$

The components of the score vector, $U_{\mathbf{\Theta}} = (U_\alpha, U_\beta)^\top$, are given by:

$$U_\alpha = \frac{\partial}{\partial\alpha}\ell(\mathbf{\Theta} \mid \mathbf{x}) = -\frac{n}{\alpha}\left(1 + \frac{2}{\alpha^2}\right) - \frac{1}{\alpha^3\,\beta}\sum_{i=1}^{n}\log x_i - \frac{\beta}{\alpha^3}\sum_{i=1}^{n}\frac{1}{\log x_i},$$

$$U_\beta = \frac{\partial}{\partial\beta}\ell(\mathbf{\Theta} \mid \mathbf{x}) = -\frac{n}{2\,\beta} + \sum_{i=1}^{n}\frac{1}{\beta - \log x_i} - \frac{1}{2\,\alpha^2\,\beta^2}\sum_{i=1}^{n}\log x_i + \frac{1}{2\,\alpha^2}\sum_{i=1}^{n}\frac{1}{\log x_i}.$$

The MLEs $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β, respectively are obtained by setting the equations $U_\alpha = U_\beta = 0$ and solving simultaneously for α and β. Thus the MLE of α can be obtained as a function of $\widehat{\beta}$:

$$\widehat{\alpha} = \left[-\frac{s}{\widehat{\beta}} - \frac{\widehat{\beta}}{r} - 2\right]^{1/2}$$

where

$$s = \frac{1}{n}\sum_{i=1}^{n}\log x_i \quad \text{and} \quad r = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\log x_i}\right]^{-1}.$$

Following the results obtained by Lemonte (2016) for the Birnbaum-Saunders distribution, we have that the elements of expected Fisher information matrix, $\mathbf{I} = [\mathbf{I}_{ij}]$, $i, j = 1, 2$, of the UBS distribution are given by:

$$\mathbf{I}_{11} = \mathrm{E}\left[-\frac{\partial^2}{\partial\alpha^2}\ell(\mathbf{\Theta} \mid \mathbf{x})\right] = \frac{2n}{\alpha^2},$$

$$\mathbf{I}_{12} = \mathbf{I}_{21} = \mathrm{E}\left[-\frac{\partial^2}{\partial\alpha\,\beta}\ell(\mathbf{\Theta} \mid \mathbf{x})\right] = 0,$$

and

$$\mathbf{I}_{22} = \mathrm{E}\left[-\frac{\partial^2}{\partial\beta^2}\ell(\mathbf{\Theta} \mid \mathbf{x})\right] = \frac{n\left[1 + \alpha\,(2\pi)^{-1/2}\,h(\alpha)\right]}{(\alpha\,\beta)^2}$$

where

$$h(\alpha) = \alpha\sqrt{\frac{\pi}{\alpha}} - \pi\,\mathrm{e}^{2/\alpha^2}\left[1 - \Phi\left(\frac{2}{\alpha}\right)\right].$$

It is clear that the UBS distribution has the same expected Fisher information matrix as BS distribution. For further details about the element $\mathbf{I}_{22}$ we refer Lemonte (2016). From the large sample theory of the maximum likelihood estimators (see, e.g. Lehmann and Casella, 1998) we have:

$$\begin{pmatrix}\widehat{\alpha}\\ \widehat{\beta}\end{pmatrix} \sim N_2\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\dfrac{\alpha^2}{2n} & 0\\ 0 & \dfrac{(\alpha\,\beta)^2}{n\left[1 + \alpha\,(2\pi)^{-1/2}\,h(\alpha)\right]}\end{pmatrix}\right].$$

It is noteworthy that α and β are orthogonal, therefore $\widehat{\alpha}$ and $\widehat{\beta}$ are asymptotically independent.

3.2 Method of Moments

Another technique fairly simple and commonly used in the parametric estimation is the method of moments. From (4), we have:

$$\mu_1 = \frac{2\,\alpha^2\beta + \sqrt{2\,\alpha^2\beta+1}+1}{4\,\alpha^2\beta+1}\exp\left[-\frac{\sqrt{2\,\alpha^2\beta+1}-1}{\alpha^2}\right],$$

$$\mu_2 = \frac{4\alpha^2\beta + \sqrt{4\,\alpha^2\beta+1}+1}{8\,\alpha^2\beta+1}\exp\left[-\frac{\sqrt{4\,\alpha^2\beta+1}-1}{\alpha^2}\right].$$

Hence, by replacing μ_1 and μ_2 for the sample first and second-order moments, i.e., $m_1 = n^{-1}\sum_{i=1}^{n} x_i$ and $m_2 = n^{-1}\sum_{i=1}^{n} x_i^2$ respectively and equating to zero, we have the following system of equations:

$$\begin{cases} m_1 - \dfrac{2\,\alpha^2\beta + \sqrt{2\,\alpha^2\beta+1}+1}{4\,\alpha^2\beta+1}\exp\left[-\dfrac{\sqrt{2\,\alpha^2\beta+1}-1}{\alpha^2}\right] = 0 \\[2ex] m_2 - \dfrac{4\alpha^2\beta + \sqrt{4\,\alpha^2\beta+1}+1}{8\,\alpha^2\beta+1}\exp\left[-\dfrac{\sqrt{4\,\alpha^2\beta+1}-1}{\alpha^2}\right] = 0. \end{cases} \tag{5}$$

It is not possible to solve analytically the system of equations (5), therefore, we have to solve it numerically.

4. Simulation Study

In this section, we perform extensive Monte Carlo simulations to evaluate the performance of the maximum likelihood estimators (MLEs) and moments estimators (MME) for the parameters of the UBS distribution. All numerical experimentation are conducted in *Ox* (Doornik, 2007) using the transformation $X = \mathrm{e}^{-Y}$, where $Y \sim \mathrm{BS}(\alpha, \beta)$, to generate pseudo-random samples from UBS distribution.

We consider the following sample sizes and parameter values: $n = 20, 30, 50$ and 100, $\alpha = 0.5, 1.0, 1.5, 2.0, 3.0,$ and 5.0 and $\beta = 0.5, 1.0, 1.5$ and 2.0. The results of the simulation studies are reported in Tables 1 and 2. For each estimator, we calculate the bias and root mean-squared error. Due to space constraint, we report the results only for $\beta = 0.5$ and 1.5. The following observations can be drawn from the Tables 1 and 2:

- All the estimators show the property of consistency i.e., the RMSE decreases as sample size increases.
- The bias of $\widehat{\alpha}$ decreases with increasing n for both the method of estimations.
- The bias of $\widehat{\beta}$ decreases with increasing n for both the method of estimations.
- The bias of $\widehat{\alpha}$ generally increases with increasing alpha for any given β and n and for both methods of estimation.

- The bias of $\widehat{\beta}$ generally increases with increasing β for any given α and n and for both methods of estimation.
- In terms of RMSE, both the methods of estimation produces smaller RMSE for $\widehat{\alpha}$ with increasing β for any given α and n.
- The bias of α of MME is smaller than bias of MLE while the bias of β of MME is greater than MLE.

Table 1. Estimated bias and root mean-squared error for α and β ($\beta = 0.5$).

		α		β	
α	n	MLE	MME	MLE	MME
0.5	20	-0.0204 (0.0811)	-0.0180 (0.0849)	0.0023 (0.0546)	0.0025 (0.0551)
	30	-0.0138 (0.0657)	-0.0122 (0.0686)	0.0014 (0.0445)	0.0015 (0.0450)
	50	-0.0086 (0.0506)	-0.0077 (0.0530)	0.0006 (0.0346)	0.0007 (0.0349)
	100	-0.0043 (0.0357)	-0.0038 (0.0374)	0.0002 (0.0242)	0.0002 (0.0245)
1.0	20	-0.0441 (0.1619)	-0.0230 (0.1844)	0.0112 (0.1029)	0.0130 (0.1120)
	30	-0.0292 (0.1318)	-0.0149 (0.1498)	0.0072 (0.0830)	0.0085 (0.0903)
	50	-0.0166 (0.1009)	-0.0082 (0.1141)	0.0044 (0.0631)	0.0053 (0.0688)
	100	-0.0079 (0.0710)	-0.0034 (0.0805)	0.0024 (0.0445)	0.0027 (0.0486)
1.5	20	-0.0660 (0.2463)	-0.0095 (0.3161)	0.0184 (0.1403)	0.0284 (0.1767)
	30	-0.0441 (0.1987)	-0.0061 (0.2518)	0.0125 (0.1118)	0.0184 (0.1388)
	50	-0.0257 (0.1520)	-0.0028 (0.1921)	0.0074 (0.0847)	0.0103 (0.1045)
	100	-0.0120 (0.1065)	-0.0007 (0.1343)	0.0037 (0.0586)	0.0053 (0.0724)
2.0	20	-0.0929 (0.3290)	0.0236 (0.4844)	0.0214 (0.1660)	0.0509 (0.2720)
	30	-0.0620 (0.2652)	0.0138 (0.3778)	0.0131 (0.1286)	0.0301 (0.1992)
	50	-0.0372 (0.2026)	0.0080 (0.2858)	0.0079 (0.0959)	0.0171 (0.1437)
	100	-0.0190 (0.1419)	0.0031 (0.1972)	0.0040 (0.0663)	0.0083 (0.0982)
3.0	20	-0.1370 (0.4979)	0.1986 (1.0341)	0.0291 (0.1923)	0.1478 (0.6063)
	30	-0.0898 (0.4021)	0.1331 (0.7870)	0.0176 (0.1467)	0.0929 (0.4169)
	50	-0.0524 (0.3077)	0.0732 (0.5515)	0.0107 (0.1082)	0.0499 (0.2642)
	100	-0.0267 (0.2147)	0.0341 (0.3630)	0.0046 (0.0737)	0.0227 (0.1604)
5.0	20	-0.2994 (0.8362)	0.3417 (1.7223)	0.0333 (0.2094)	0.2727 (0.9763)
	30	-0.1777 (0.6657)	0.4010 (1.5608)	0.0204 (0.1566)	0.2163 (0.7973)
	50	-0.0900 (0.5077)	0.3651 (1.3161)	0.0119 (0.1144)	0.1499 (0.5942)
	100	-0.0398 (0.3564)	0.2056 (0.8891)	0.0052 (0.0776)	0.0782 (0.3540)

5. Applications

In this section, we analyze two real data sets to illustrate the potentiality of the proposed distribution. For the sake of comparison, we consider the Beta and Kumaraswamy (Kum) distributions, since they are widely used to modeling data with bounded domain. Their corresponding probability density functions are given by:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} \quad \text{and} \quad f(x) = \alpha\,\beta\, x^{\alpha-1}\,(1-x^{\alpha})^{\beta-1}$$

respectively, where $\alpha > 0$ and $\beta > 0$ are shape parameters.

For the two data sets, we compare the fits of the UBS distribution with Beta and Kumaraswamy distributions to illustrate the potentiality of the UBS model. We estimate the unknown parameters of the distributions by maximum likelihood method. For model

Table 2. Estimated bias and root mean-squared error for α and β ($\beta = 1.5$).

		α		β	
α	n	MLE	MME	MLE	MME
0.5	20	-0.0201 (0.0810)	-0.0159 (0.0884)	0.0089 (0.1640)	0.0101 (0.1680)
	30	-0.0137 (0.0660)	-0.0109 (0.0718)	0.0052 (0.1337)	0.0060 (0.1372)
	50	-0.0082 (0.0504)	-0.0066 (0.0550)	0.0032 (0.1026)	0.0036 (0.1052)
	100	-0.0039 (0.0355)	-0.0032 (0.0388)	0.0022 (0.0727)	0.0023 (0.0746)
1.0	20	-0.0421 (0.1625)	-0.0135 (0.2155)	0.0288 (0.3091)	0.0556 (0.3886)
	30	-0.0275 (0.1317)	-0.0088 (0.1711)	0.0178 (0.2482)	0.0345 (0.3027)
	50	-0.0162 (0.1007)	-0.0050 (0.1297)	0.0099 (0.1896)	0.0199 (0.2270)
	100	-0.0080 (0.0705)	-0.0021 (0.0898)	0.0040 (0.1330)	0.0097 (0.1564)
1.5	20	-0.0647 (0.2435)	0.0210 (0.4103)	0.0571 (0.4204)	0.1704 (0.7793)
	30	-0.0431 (0.1981)	0.0141 (0.3218)	0.0373 (0.3331)	0.1076 (0.5672)
	50	-0.0271 (0.1527)	0.0047 (0.2372)	0.0203 (0.2511)	0.0561 (0.3856)
	100	-0.0125 (0.1071)	0.0039 (0.1630)	0.0102 (0.1751)	0.0280 (0.2553)
2.0	20	-0.0916 (0.3265)	0.0720 (0.6644)	0.0646 (0.4921)	0.2936 (1.1686)
	30	-0.0583 (0.2648)	0.0649 (0.5422)	0.0453 (0.3899)	0.2249 (0.9491)
	50	-0.0327 (0.2035)	0.0417 (0.3929)	0.0268 (0.2902)	0.1316 (0.6390)
	100	-0.0165 (0.1428)	0.0201 (0.2588)	0.0137 (0.2020)	0.0617 (0.3856)
3.0	20	-0.1520 (0.5018)	0.1226 (1.1448)	0.0665 (0.5667)	0.3649 (1.5585)
	30	-0.0985 (0.4028)	0.1521 (1.0071)	0.0471 (0.4360)	0.3710 (1.4353)
	50	-0.0574 (0.3066)	0.1419 (0.8144)	0.0314 (0.3261)	0.3054 (1.1752)
	100	-0.0282 (0.2136)	0.0879 (0.5573)	0.0167 (0.2235)	0.1710 (0.7638)
5.0	20	-0.2835 (0.8487)	-0.1528 (1.6323)	0.0618 (0.6049)	0.0948 (1.5696)
	30	-0.1854 (0.6770)	-0.0055 (1.5280)	0.0357 (0.4559)	0.1810 (1.5060)
	50	-0.1085 (0.5142)	0.1422 (1.4175)	0.0230 (0.3395)	0.2795 (1.4511)
	100	-0.0493 (0.3581)	0.2338 (1.2414)	0.0120 (0.2301)	0.3152 (1.2757)

comparison, we consider only Kolmogorov-Smirnov (K-S) statistic with its p-values and Cramér-von-Mises (CvM) statistic. A model which yields a smaller values of the K-S statistic and CvM statistic are said to be a better fit.

First Application

The first data set corresponds to the monthly water capacity data from the Shasta reservoir in California, USA and were taken for the month of February from 1991 to 2010, http://cdec.water.ca.gov/reservoir.html (Nadar et al., 2013). The data has 20 observations and were used by Nadar et al. (2013) and Wang et al. (2017) to fit the Kumaraswamy distribution.

The results reported in Table 3 indicates that all the aforementioned distributions fitted the data, however the UBS distribution is the best, since it has the lowest values of K-S and CvM statistics. Also, the Probability-Probability plots displays in Figure 3 supports the conclusion that the UBS provides the best fit among Beta and Kumaraswamy distributions.

Table 3. Summary of fitted distributions — Water capacity data.

Dist.	$\widehat{\alpha}$	†C.I.	$\widehat{\beta}$	†C.I.	K-S	CvM
UBS	0.5781	(0.3990, 0.7573)	0.3052	(0.2311,0.3793)	0.2172 (0.2615)	0.2370 (0.2064)
Beta	7.3154	(2.7722, 11.8586)	2.9098	(1.1940,4.6256)	0.2359 (0.1834)	0.2860 (0.1479)
Kum	6.3476	(3.2950, 9.4002)	4.4893	(0.4893,8.4894)	0.2209 (0.2447)	0.2505 (0.1880)

†: 95% confidence interval.

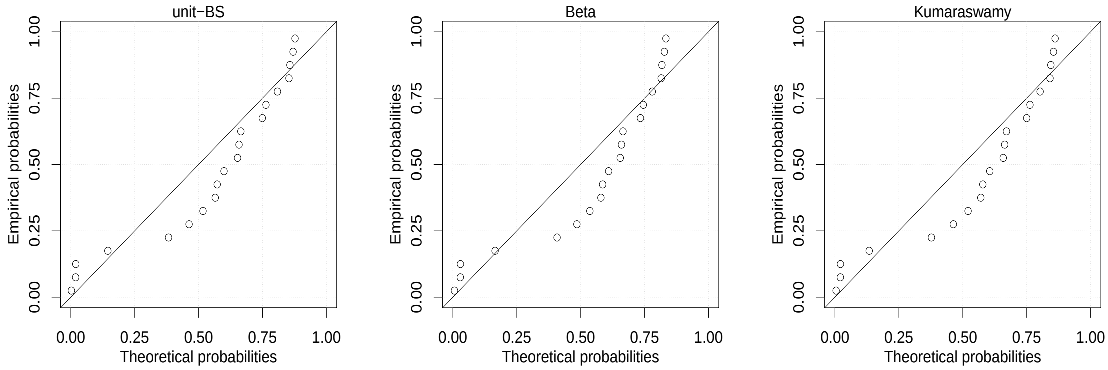

Figure 3. PP-Plots of the fitted distributions — Water capacity data.

Second Application

The second data set represents the total milk production in the first birth of 107 cows from SINDI race. This data set is taken from Cordeiro and dos Santos (2012).

Table 4 presents the results of the fitted distributions. From Table 4 we see that the UBS has the lowest values of K-S and CvM statistics. Therefore, the UBS is the best choice among Beta and Kumaraswamy distributions. This is also supported by the P-P-Plots in Figure 4.

Table 4. Summary of fitted distributions — Milk production data.

Dist.	$\widehat{\alpha}$	†C.I.	$\widehat{\beta}$	†C.I.	K-S	CvM
UBS	0.6546	(0.5669, 0.7423)	0.7408	(0.6538,0.8278)	0.0711 (0.6524)	0.0964 (0.6039)
Beta	2.4126	(1.7962, 3.0290)	2.8297	(2.0959,3.5636)	0.0910 (0.3384)	0.2282 (0.2190)
Kum	2.1949	(1.7590, 2.6307)	3.4363	(2.2956,4.5770)	0.0762 (0.5626)	0.1521 (0.3836)

†: 95% confidence interval.

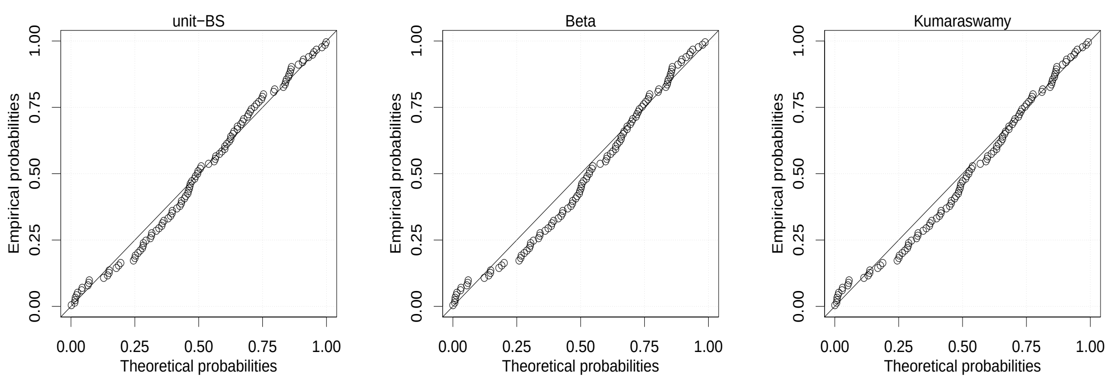

Figure 4. PP-Plots of the fitted distributions — Milk production data.

6. Conclusion

In this article, a new probability density function with bounded domain is proposed which can serve as an alternatives to Beta and Kumaraswamy distributions having two shape parameters. The estimation of model parameters are obtained by the method of maximum likelihood and moments. Two applications to real data sets are presented as an illustration of the potentiality of the new model as compared to Beta and Kumaraswamy models. After comparing the values of two popular goodness-of-fit statistics, we may say that our model UBS is better as compared to Beta and Kumaraswamy distributions for these two data sets. We expect the utility of the newly proposed model in different fields especially in life-time and reliability when hazard rate is increasing or bathtub shaped.

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