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THE UNIT-WEIBULL DISTRIBUTION AND ASSOCIATED INFERENCE

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SUMMARY

Distributions such as the Simplex, Johnson S_B, unit-Gamma and unit-Logistic, each with support on the unit interval (0,1), are formulated using appropriate transformation of random variables following inverse-Gaussian mixture, Normal, Gamma and Logistic distributions, respectively. These distributions can serve the same purpose of the Beta and Kumaraswamy distributions. In this paper, we propose a new two-parameter unit-Weibull distribution which is also useful for modeling data on the unit interval (0,1). Some properties of this new distribution are studied. Monte Carlo simulations reveal that the maximum likelihood estimators are nearly unbiased and consistent. The potential of this new distribution is illustrated using two real data sets.

Keywords and phrases: Beta distribution, Maximum likelihood estimation, Monte Carlo simulation, unit-Gamma distribution, Weibull distribution.

2010 Mathematics Subject Classification: 60E05, 62F10.

1 Introduction

Although the Beta distribution is flexible and has been the most used to modeled data on bounded domain, in the last years several works have been proposed new distributions on a unit interval. We can mention the following: the Johnson S_B distribution [18], the Johnson S_B' distribution [19], the Topp-Leone distribution [32], the unit-Gamma distribution [17, 30], the Kumaraswamy distribution [21], the Arcsine distribution [2], the unit-Logistic distribution [31], the McDonald's generalized Beta type I distribution [23], the Simplex distribution [3], the reflected Generalized Topp-Leone distribution [33], the Beta power distribution [10], the McDonald Arcsine distribution [11], the Log-Lindley distribution [16], the exponentiated Kumaraswamy distribution [22], the exponentiated Topp-Leone distribution [25], the Marshall-Olkin extended Kumaraswamy [7], the reflected generalized Topp-Leone power series distribution [9], the transmuted Kumaraswamy distribution [29], the size biased Kumaraswamy distribution [28] and the extended Arcsine distribution [12]. It should be pointed that the majority of these distributions have more than two parameters, which considering limited amount of data, may produce inaccurate estimates.

Here, following [17] and [30], we propose a new distribution with support on the unit-interval $(0,1)$, which arises from a certain transformation on the two-parameter Weibull distribution [35] with probability density function (p.d.f.)

$$g(y;\alpha,\beta)=\alpha\,\beta\,y^{\beta-1}\,\mathrm{e}^{-\alpha\,y^{\beta}},\qquad y>0,\quad \alpha,\beta>0, \tag{1.1}$$

where α and β are the scale and shape parameters, respectively.

Using the transformation $X=\mathrm{e}^{-Y}$, we have a new distribution on $(0,1)$, which we refer to as unit-Weibull (UW) distribution. Its cumulative distribution function is given by

$$F(x;\alpha,\beta)=\exp\left[-\alpha\,(-\log x)^{\beta}\right],\qquad 0<x<1,\quad \alpha,\beta>0, \tag{1.2}$$

and the corresponding p.d.f. is

$$f(x;\alpha,\beta)=\frac{1}{x}\,\alpha\,\beta\,(-\log x)^{\beta-1}\,\exp\left[-\alpha\,(-\log x)^{\beta}\right],\qquad 0<x<1,\quad \alpha,\beta>0. \tag{1.3}$$

Note that α is no longer a scale parameter, since $f(\alpha x;\alpha,\beta)\neq\frac{1}{\alpha}f(x;1,\beta)$. Special cases of the UW distributions include: the standard uniform distribution over the interval (0,1) ($\alpha=\beta=1$), the power function distribution ($\beta=1$) and the unit-Rayleigh distribution ($\beta=2$). Therefore, the new distribution has connection with some well known distributions, and hence, it can be very useful in many practical situations. Figure 1 shows some possible shapes of the p.d.f. of the UW distribution for selected values of the parameters α and β.

The purpose of this paper are to introduce and study some properties of the UW distribution. In Section 2, we present some features of the UW distribution. We discuss the maximum likelihood estimation and inference of the model parameters in Section 3, where we also derived explicit expressions for the expected Fisher information matrix. Monte Carlo simulations are conducted in Section 4 in order to study some properties of the maximum likelihood estimators and to evaluate the coverage probability of asymptotic confidence intervals. Section 5 shows the comparison between the new proposed distribution and some other distributions using two real data sets. Finally, some concluded remarks are given in Section 6.

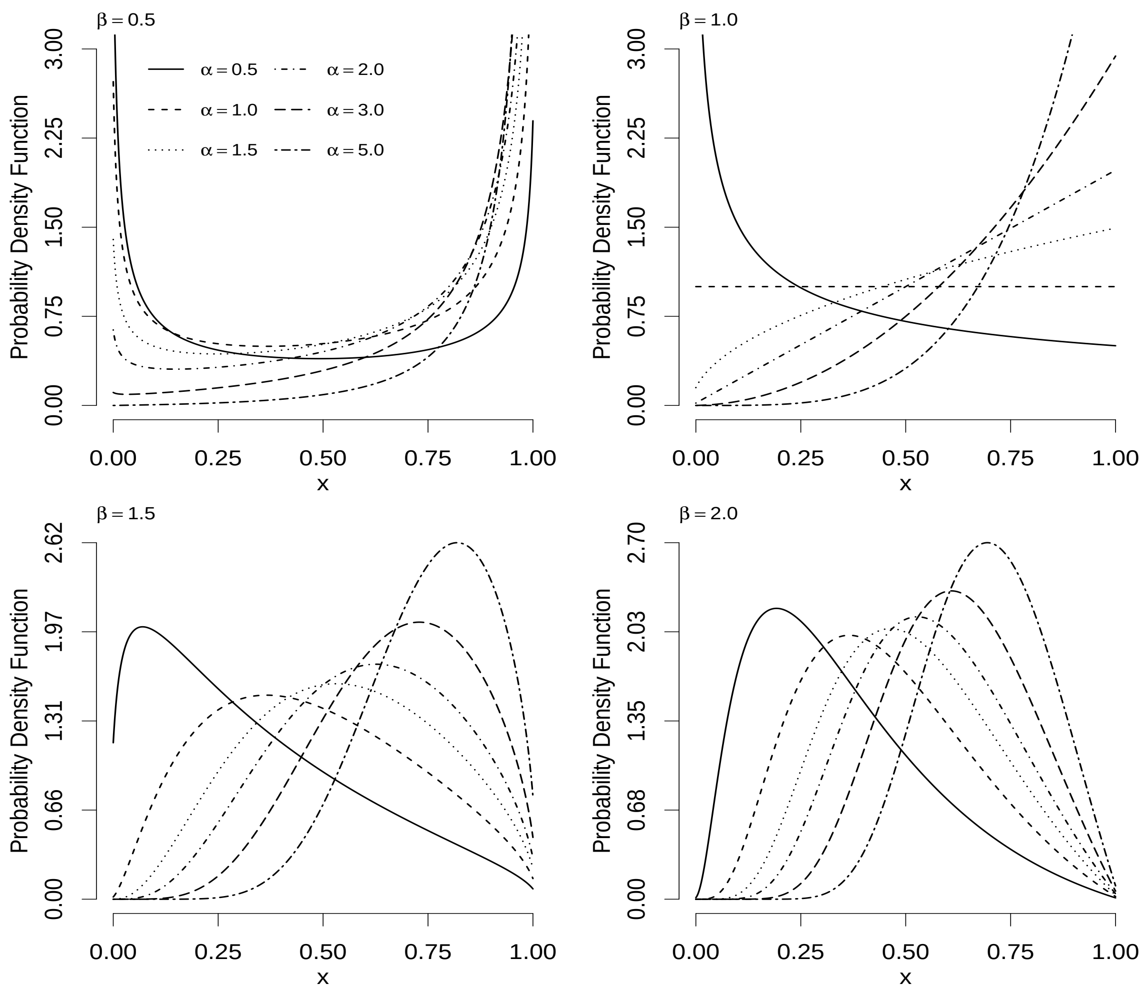

Figure 1: Probability density function of the UW distribution for selected values of α and β.

Note that, unlike the Beta distribution, the proposed model has closed form expression for the quantile function. This fact can be used to introduce a quantile regression model which may be a more flexible alternative to the classical Beta regression model [8, 15]. As discussed in the statistical literature [20, 36, 24, 27, 4] the quantile regression analysis has been used in several contexts and its main advantage when compared with the conditional-mean regressions, such as Beta and Simplex, is that it provides a complete view of the conditional distribution by studying distinct quantiles. By employing quantile regression such as conditional-median regressions, practitioners will have a more robust model for outliers than the usual Beta regression. Another advantage lies on the fact that if the conditional dependent variable is skewed, the median may be a more appropriate when compared with the mean. Since the main goal of this paper is to introduce and study some of properties of the UW distribution, the quantile regression issue is not addressed throughout the text.

2 Statistical Properties

In this section, we explore some statistical properties of the proposed UW distribution.

2.1 Hazard rate function

The hazard rate function of the UW distribution is given by

$$h(x;\alpha,\beta)=\frac{f(x;\alpha,\beta)}{1-F(x;\alpha,\beta)}=\frac{\alpha\beta\;(-\log x)^{\beta-1}\;\exp\left[-\alpha\,(-\log x)^{\beta}\right]}{x\,(1-\exp\left[-\alpha\,(-\log x)^{\beta}\right])},\qquad 0<x<1. \tag{2.1}$$

Figure 2 shows some possible shapes of the hazard rate function of the UW distribution for selected values of the parameters α and β. Figure 2 shows increasing or bathtub shapes of the hazard rate function of the UW distribution. These shapes are also similar to the shapes of the Beta distribution (see Ghitany, 2004).

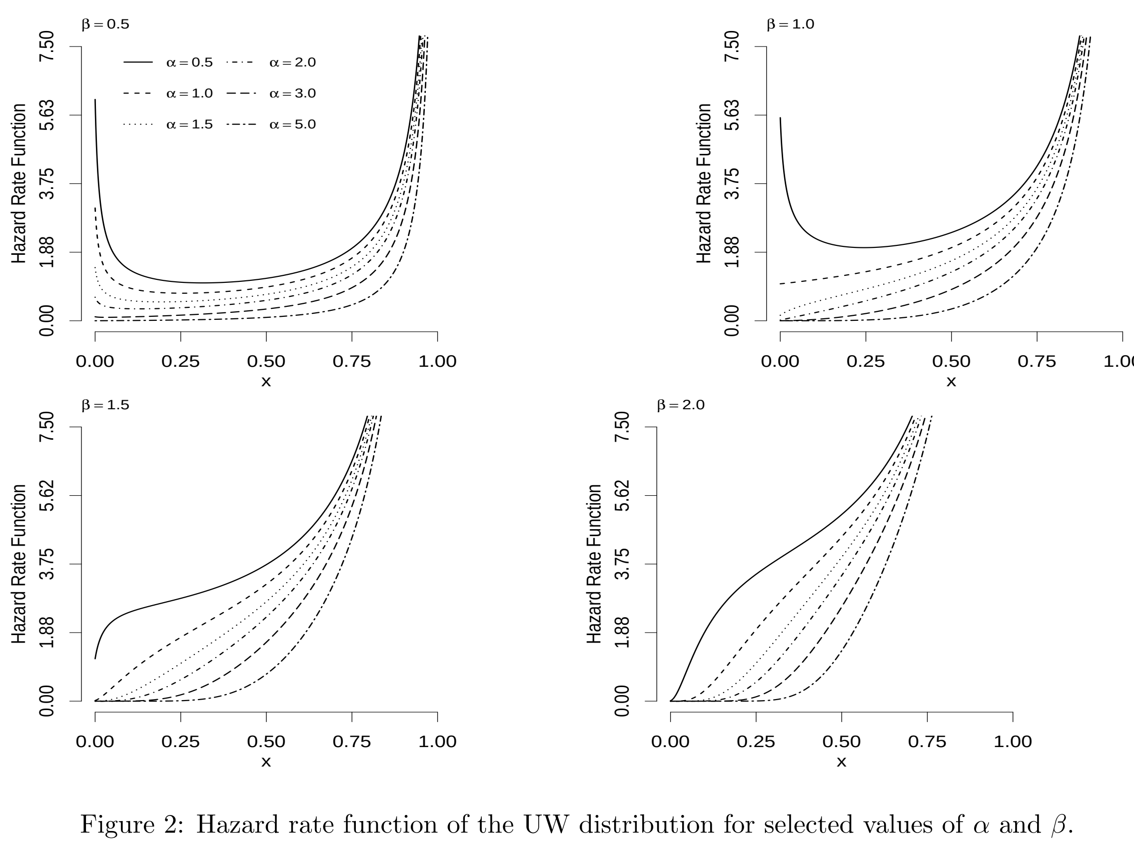

Figure 2: Hazard rate function of the UW distribution for selected values of α and β.

2.2 Moments and associated measures

The rth raw moment of the UW distribution is given by

$$\mu'_r = E(X^r) = E(e^{-rY}) = M_Y(-r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ \alpha^{n/\beta}}\ \Gamma\left(\frac{n}{\beta}+1\right).$$

The skewness and kurtosis measures can be obtained from the expressions

$$skewness = \frac{\mu'_3 - 3\mu'_2\mu + \mu^3}{\sigma^3},$$

$$kurtosis = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{\sigma^4},$$

upon substituting for the raw moments.

However, for the special case $\beta = 1$, i.e. the power function distribution, we have

$$\mu'_r = E(X^r) = \frac{\alpha}{r+\alpha}, \qquad r = 1, 2, \ldots.$$

In this case, the mean, variance, skewness and kurtosis, respectively, are given by

$$\mu = \frac{\alpha}{1+\alpha}, \qquad \sigma^2 = \frac{\alpha}{(1+\alpha)^2(2+\alpha)},$$

$$skewness = \frac{2(1-\alpha)}{(2+\alpha)}\sqrt{1+\frac{2}{\alpha}} \qquad \text{and} \qquad kurtosis = \frac{3(2+\alpha)(2-\alpha+3\alpha^2)}{\alpha(3+\alpha)(4+\alpha)}.$$

Note that, in this case, the skewness can be negative, zero, positive when $\alpha < 1, \alpha = 1, \alpha > 1$, respectively.

Setting $\alpha = 1$ in the last expressions, i.e., the standard uniform distribution, we obtain

$$\mu = \frac{1}{2}, \qquad \sigma^2 = \frac{1}{12}, \qquad skewness = 0 \text{ and } kurtosis = \frac{9}{5}.$$

Similarly, for the special case $\beta = 2$, i.e., the unit-Rayleigh distribution, we have

$$\mu'_r = E(X^r) = 1 - \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\ r\ e^{r^2/(4\alpha)}\ \text{erfc}\left(\frac{r}{2\sqrt{\alpha}}\right), \qquad r = 1, 2, \ldots,$$

where

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-x^2}\ dx, \qquad z > 0,$$

is the complementary error function. In this case, the mean and variance of the UW distribution, respectively, are

$$\mu = 1 - \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\ e^{1/(4\alpha)}\ \text{erfc}\left(\frac{1}{2\sqrt{\alpha}}\right),$$

$$\sigma^2 = 1 - \frac{\sqrt{\pi}}{\sqrt{\alpha}}\ e^{1/\alpha}\ \text{erfc}\left(\frac{1}{\sqrt{\alpha}}\right) - \left[1 - \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\ e^{1/(4\alpha)}\ \text{erfc}\left(\frac{1}{2\sqrt{\alpha}}\right)\right]^2.$$

Figure 3 shows the mean, variance, skewness and kurtosis of the UW distribution as a function of β for various values of α. This figure shows that the skewness can be negative for values $\beta \neq 1$ which means that for modeling negatively skewed data, the UW distribution will be a useful model.

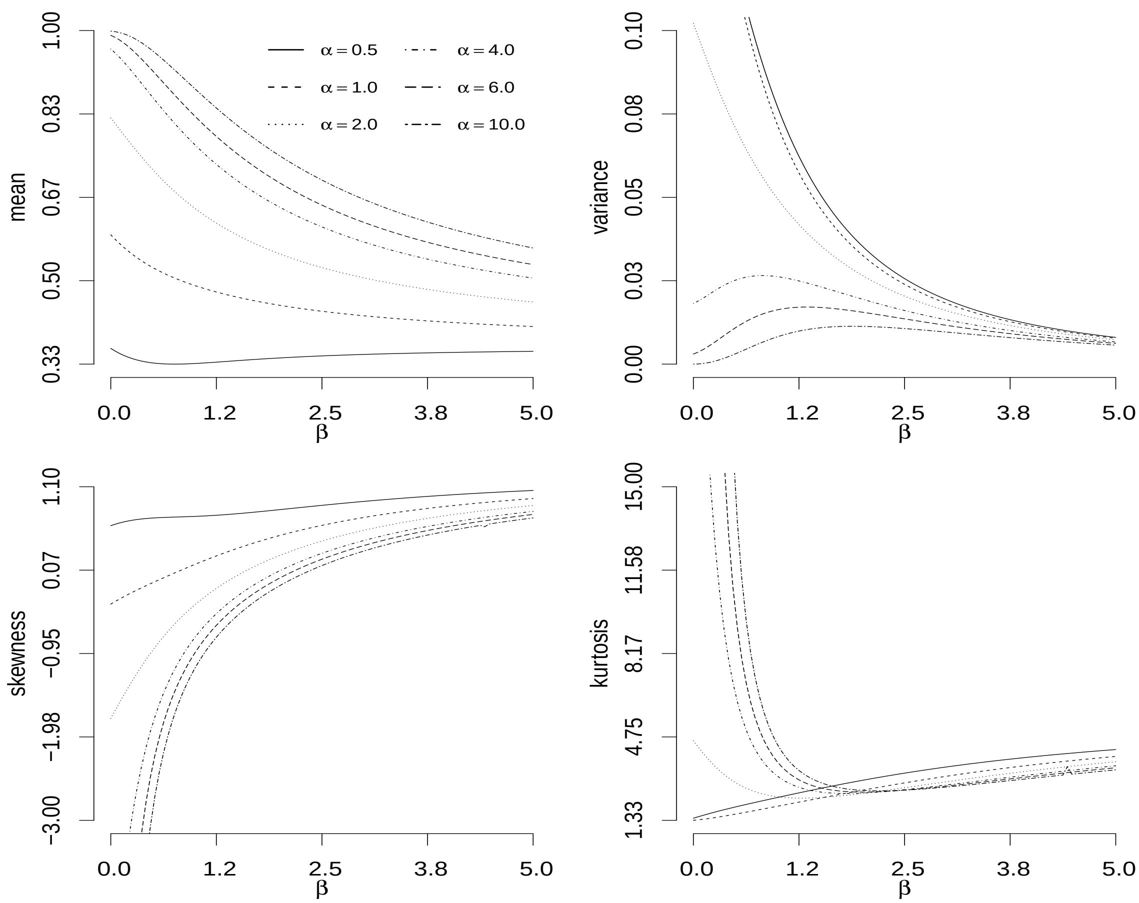

Figure 3: Mean, variance, skewness and kurtosis of the UW distribution for selected values of α and β.

2.3 Quantile function and associated measures

The quantile function of the UW distribution is given by

$$Q(p) = \exp\left[-\left(-\frac{\log p}{\alpha}\right)^{\frac{1}{\beta}}\right], \qquad 0 < p < 1. \tag{2.2}$$

The special cases $\alpha = \beta = 1$, $\beta = 1$ and $\beta = 2$, respectively, give $Q(p) = p$, $Q(p) = p^{1/\alpha}$ and $Q(p) = \exp(-\sqrt{-\log(p)/\alpha})$. The quartiles of the UW distribution, as well as the special cases, are obtained by setting $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively.

3 Maximum Likelihood Estimation

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample of size n from the UW distribution with p.d.f. (1.3). Then, the log-likelihood function of $\boldsymbol{\theta} = (\alpha, \beta)$ is given by

$$\begin{aligned} \ell(\boldsymbol{\theta}; \mathbf{x}) &= \sum_{i=0}^{n} \log f(x_i; \boldsymbol{\theta}) \\ &= n(\log\alpha + \log\beta) - \sum_{i=1}^{n} \log x_i + (\beta - 1) \sum_{i=1}^{n} \log(-\log x_i) - \alpha \sum_{i=1}^{n} (-\log x_i)^{\beta}. \end{aligned} \quad (3.1)$$

likelihood estimate $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained by solving the non-linear equations

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} (-\log x_i)^{\beta} = 0, \quad (3.2)$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(-\log x_i) - \alpha \sum_{i=1}^{n} (-\log x_i)^{\beta} \log(-\log x_i) = 0. \quad (3.3)$$

Equation (3.2) can be solved algebraically for α, giving $\widehat{\alpha}(\beta) = \dfrac{n}{\sum_{i=1}^{n} (-\log x_i)^{\beta}}$.

To obtain $\widehat{\beta}$, we substitute $\widehat{\alpha}(\beta)$ into (3.3) and solve for β. We have

$$g(\beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \log(-\log x_i) - \frac{n \sum_{i=1}^{n} (-\log x_i)^{\beta} \log(-\log x_i)}{\sum_{i=1}^{n} (-\log x_i)^{\beta}}. \quad (3.4)$$

Equation (3.4) can be solved numerically using, for example, Brent's method [6] available in software `R` [26] through the `uniroot` function. This method has the advantage that it does not require computation of the derivative $g'(\beta)$ and initial guess for β can be provided as an interval.

Note that (1.2) satisfies $\log[-\log F(x_i; \alpha, \beta)] = \log\alpha + \beta \log(-\log x_i)$, for $i = 1, \ldots, n$. Thus a plot of $\log[-\log \widehat{F}(x_{(i)})]$ versus $\log(-\log x_{(i)})$ would be roughly linear if a UW distribution is appropriate, where $\widehat{F}(x_{(i)})$ is the empirical distribution function at the ordered observed value $x_{(i)}$. In addition, when the plot is approximately linear, one can obtain empirical estimates of α and β by fitting a straight line. This empirical estimate of β can be used as initial guess to solve numerically equation (3.4).

The expected Fisher information matrix of $\boldsymbol{\theta} = (\alpha, \beta)$ based on a single observation is given by

$$\begin{aligned} \mathbf{I}(\boldsymbol{\theta}) = [I_{ij}] &= \left[-E\left(\frac{\partial^2 \log f(x_i; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j} \right) \right], \qquad i, j = 1, 2, \\ &= \begin{pmatrix} \dfrac{1}{\alpha} & \frac{1}{\alpha\beta}(1 - \gamma - \log\alpha) \\ \frac{1}{\alpha\beta}(1 - \gamma - \log\alpha) & \frac{1}{6\beta^2}\left[\pi^2 + 6(1 - \gamma - \log\alpha)^2\right] \end{pmatrix}, \end{aligned} \quad (3.5)$$

where $\pi \simeq 3.141593$ and $\gamma \simeq 0.577216$ is the Euler's constant.

Under mild regularity conditions (see Lehmann and Casella, 1998, pp. 461–463), the asymptotic distribution of the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is such that

$$\sqrt{n}\ (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\mathrm{D}} N(0, \mathbf{I}(\boldsymbol{\theta})),$$

where $\xrightarrow{\mathrm{D}}$ denotes convergence in distribution and $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the inverse of the matrix $\mathbf{I}(\boldsymbol{\theta})$, with

$$\mathbf{I}^{-1}(\boldsymbol{\theta}) = [\sigma_{ij}] = \begin{pmatrix} \frac{\alpha^2}{\pi^2}\left[\pi^2 + 6(1-\gamma-\log\alpha)^2\right] & -\dfrac{6\alpha\beta}{\pi^2}\ (1-\gamma-\log\alpha) \\ & \\ -\dfrac{6\alpha\beta}{\pi^2}\ (1-\gamma-\log\alpha) & \dfrac{6\beta^2}{\pi^2} \end{pmatrix}.$$

The large-sample $100(1-\delta)\%$ confidence intervals for α and β, respectively, are given by

$$\widehat{\alpha} \pm z_{\delta/2}\ \frac{\sqrt{\widehat{\sigma}_{11}}}{\sqrt{n}} \qquad \text{and} \qquad \widehat{\beta} \pm z_{\delta/2}\ \frac{\sqrt{\widehat{\sigma}_{22}}}{\sqrt{n}},$$

where $\widehat{\sigma}_{11}$ and $\widehat{\sigma}_{22}$, respectively, are the estimated asymptotic variances of the maximum likelihood estimators $\widehat{\alpha}$, $\widehat{\beta}$ and z_q is the upper q-th quantile of the standard normal distribution.

Although not considered in this paper, it is important to note that for a Bayesian analysis we can use the Jeffreys invariant prior [5] for $\boldsymbol{\theta}$, given by $\pi(\boldsymbol{\theta}) \propto \sqrt{|I(\boldsymbol{\theta})|}$ where $|I(\boldsymbol{\theta})|$ is the determinant of (10). Two alternative prior joint distributions for α and β can be found, for example, in [1].

4 Simulation Studies

In this section, we carry out Monte Carlo simulations to study the finite-sample behavior of the MLEs and the asymptotic confidence intervals for the parameters of UW distribution. The evaluation was performed based on the estimated bias, the estimated root mean-squared error (RMSE) and the coverage probabilities. We set the samples sizes $n = 10, 20, 50, 100, 200$ and 500, $\alpha = 0.5, 1.0, 1.5, 2.0, 3.0$ and 5.0 and $\beta = 0.5, 1.0, 1.5$ and 2.0. For each combination of n, α and β, we generated random samples from $Y \sim$ Weibull(α, β) and apply transformation e^{-Y}. The number of Monte Carlo replications was fixed at $M = 10,000$. All simulations were conducted in Ox Console [13], using the `MaxBFGS` function to obtain the maximum likelihood estimates of α and β. The results are reported in Tables 1–4.

Table 1: Estimated bias, root mean-squared and coverage probability α and β ($\beta = 0.5$).

		Bias		RMSE		$CP_{90\%}$		$CP_{95\%}$	
α	n	α	β	α	β	α	β	α	β
0.5	10	0.0150	0.0833	0.2458	0.1907	0.8465	0.8885	0.9201	0.9426
	20	0.0044	0.0366	0.1533	0.1077	0.8784	0.8941	0.9372	0.9456
	50	0.0010	0.0143	0.0949	0.0605	0.8930	0.8954	0.9450	0.9463
	100	0.0008	0.0064	0.0671	0.0405	0.8920	0.9023	0.9444	0.9499
	200	0.0011	0.0029	0.0477	0.0282	0.8935	0.9020	0.9452	0.9497
	500	0.0002	0.0014	0.0296	0.0174	0.9004	0.9029	0.9489	0.9502
1.0	10	0.1080	0.0855	0.5083	0.1954	0.8758	0.8843	0.9358	0.9404
	20	0.0402	0.0384	0.2765	0.1090	0.8825	0.8932	0.9394	0.9451
	50	0.0119	0.0141	0.1568	0.0600	0.8940	0.8987	0.9455	0.9480
	100	0.0059	0.0066	0.1076	0.0408	0.8971	0.8953	0.9472	0.9462
	200	0.0039	0.0035	0.0754	0.0284	0.8998	0.8945	0.9486	0.9458
	500	0.0017	0.0014	0.0472	0.0176	0.8989	0.8995	0.9481	0.9484
1.5	10	0.2618	0.0836	0.8520	0.1909	0.9269	0.8868	0.9628	0.9417
	20	0.1092	0.0378	0.4475	0.1086	0.8961	0.8930	0.9466	0.9450
	50	0.0392	0.0149	0.2371	0.0609	0.8949	0.8955	0.9460	0.9463
	100	0.0180	0.0072	0.1584	0.0407	0.8984	0.8979	0.9478	0.9476
	200	0.0091	0.0035	0.1088	0.0280	0.8954	0.8973	0.9463	0.9473
	500	0.0032	0.0015	0.0675	0.0176	0.8996	0.8947	0.9485	0.9459
2.0	10	0.4738	0.0823	1.2696	0.1887	0.9436	0.8865	0.9714	0.9415
	20	0.1993	0.0376	0.6519	0.1104	0.9129	0.8917	0.9554	0.9443
	50	0.0663	0.0142	0.3275	0.0602	0.9051	0.8982	0.9514	0.9477
	100	0.0290	0.0063	0.2180	0.0406	0.9003	0.8989	0.9488	0.9481
	200	0.0119	0.0029	0.1491	0.0280	0.8988	0.9012	0.9481	0.9493
	500	0.0036	0.0011	0.0922	0.0175	0.9048	0.8988	0.9512	0.9481
3.0	10	0.7528	0.0687	1.7562	0.1655	0.9523	0.9113	0.9759	0.9546
	20	0.4328	0.0375	1.1766	0.1078	0.9357	0.9008	0.9673	0.9491
	50	0.1555	0.0141	0.5872	0.0607	0.9134	0.8923	0.9557	0.9446
	100	0.0708	0.0066	0.3707	0.0407	0.9066	0.8975	0.9522	0.9474
	200	0.0358	0.0032	0.2515	0.0283	0.9027	0.8959	0.9501	0.9465
	500	0.0154	0.0013	0.1557	0.0176	0.9022	0.8981	0.9498	0.9477
5.0	10	0.5698	0.0329	1.9063	0.1254	0.9412	0.9367	0.9702	0.9678
	20	0.5397	0.0241	1.6698	0.0916	0.9464	0.9207	0.9728	0.9595
	50	0.3418	0.0137	1.1562	0.0592	0.9251	0.9026	0.9618	0.9501
	100	0.1743	0.0070	0.7669	0.0405	0.9049	0.8994	0.9513	0.9484
	200	0.0900	0.0036	0.5160	0.0285	0.9024	0.8956	0.9499	0.9464
	500	0.0313	0.0014	0.3125	0.0176	0.9053	0.9020	0.9515	0.9497

Table 2: Estimated bias, root mean-squared and coverage probability α and β ($\beta = 1.0$).

		Bias		RMSE		$CP_{90\%}$		$CP_{95\%}$	
α	n	α	β	α	β	α	β	α	β
0.5	10	0.0098	0.1691	0.2337	0.3891	0.8501	0.8842	0.9220	0.9403
	20	0.0004	0.0783	0.1523	0.2208	0.8765	0.8969	0.9362	0.9470
	50	-0.0004	0.0290	0.0939	0.1220	0.8960	0.8995	0.9466	0.9484
	100	0.0004	0.0140	0.0659	0.0821	0.8989	0.9006	0.9481	0.9490
	200	0.0003	0.0064	0.0468	0.0567	0.8966	0.9008	0.9469	0.9491
	500	0.0004	0.0025	0.0295	0.0352	0.9030	0.9000	0.9503	0.9487
1.0	10	0.1060	0.1751	0.5173	0.3890	0.8740	0.8853	0.9349	0.9409
	20	0.0380	0.0777	0.2751	0.2187	0.8849	0.8954	0.9407	0.9463
	50	0.0125	0.0301	0.1570	0.1212	0.8988	0.8990	0.9481	0.9482
	100	0.0050	0.0149	0.1071	0.0819	0.9012	0.8966	0.9493	0.9469
	200	0.0033	0.0068	0.0746	0.0563	0.9027	0.9001	0.9501	0.9487
	500	0.0007	0.0029	0.0466	0.0353	0.9052	0.8985	0.9514	0.9479
1.5	10	0.2610	0.1754	0.8426	0.3959	0.9261	0.8829	0.9623	0.9396
	20	0.1039	0.0776	0.4367	0.2227	0.9012	0.8891	0.9493	0.9429
	50	0.0368	0.0288	0.2344	0.1212	0.9005	0.8948	0.9489	0.9459
	100	0.0180	0.0132	0.1560	0.0813	0.9015	0.9024	0.9495	0.9499
	200	0.0080	0.0063	0.1068	0.0568	0.9052	0.8933	0.9514	0.9451
	500	0.0031	0.0024	0.0670	0.0355	0.9023	0.8951	0.9499	0.9461
2.0	10	0.4580	0.1584	1.2515	0.3679	0.9378	0.8901	0.9684	0.9434
	20	0.2024	0.0743	0.6800	0.2176	0.9075	0.8936	0.9526	0.9453
	50	0.0704	0.0296	0.3330	0.1222	0.9039	0.8935	0.9507	0.9453
	100	0.0324	0.0149	0.2144	0.0813	0.9055	0.9035	0.9516	0.9505
	200	0.0156	0.0086	0.1478	0.0559	0.9014	0.9044	0.9494	0.9510
	500	0.0068	0.0034	0.0923	0.0347	0.9012	0.9070	0.9493	0.9524
3.0	10	0.7296	0.1363	1.7602	0.3359	0.9475	0.9092	0.9734	0.9535
	20	0.4091	0.0712	1.1710	0.2143	0.9337	0.9015	0.9663	0.9495
	50	0.1440	0.0264	0.5767	0.1188	0.9045	0.8979	0.9511	0.9476
	100	0.0686	0.0133	0.3705	0.0820	0.9020	0.8943	0.9497	0.9457
	200	0.0343	0.0063	0.2510	0.0563	0.9021	0.8986	0.9498	0.9479
	500	0.0121	0.0026	0.1544	0.0349	0.9014	0.9022	0.9494	0.9498
5.0	10	0.5237	0.0614	1.9039	0.2484	0.9379	0.9396	0.9685	0.9693
	20	0.5298	0.0472	1.6565	0.1820	0.9399	0.9253	0.9695	0.9619
	50	0.3347	0.0272	1.1703	0.1193	0.9184	0.8989	0.9583	0.9481
	100	0.1618	0.0138	0.7705	0.0820	0.9040	0.8988	0.9508	0.9480
	200	0.0818	0.0072	0.5189	0.0567	0.9000	0.8959	0.9487	0.9465
	500	0.0277	0.0027	0.3129	0.0354	0.8992	0.8966	0.9483	0.9469

Table 3: Estimated bias, root mean-squared and coverage probability α and β ($\beta = 1.5$).

		Bias		RMSE		$CP_{90\%}$		$CP_{95\%}$	
α	n	α	β	α	β	α	β	α	β
0.5	10	0.0098	0.2541	0.2450	0.5860	0.8376	0.8851	0.9152	0.9408
	20	0.0002	0.1145	0.1566	0.3309	0.8651	0.8885	0.9301	0.9426
	50	0.0003	0.0399	0.0959	0.1828	0.8869	0.8910	0.9418	0.9439
	100	-0.0001	0.0198	0.0667	0.1214	0.8977	0.9020	0.9475	0.9497
	200	-0.0002	0.0098	0.0470	0.0846	0.8988	0.9008	0.9481	0.9491
	500	0.0003	0.0040	0.0299	0.0533	0.8948	0.8969	0.9459	0.9470
1.0	10	0.1027	0.2559	0.4828	0.5822	0.8807	0.8875	0.9385	0.9421
	20	0.0396	0.1162	0.2759	0.3282	0.8868	0.8929	0.9417	0.9449
	50	0.0147	0.0398	0.1556	0.1806	0.8971	0.8966	0.9472	0.9469
	100	0.0084	0.0184	0.1074	0.1213	0.9008	0.9022	0.9491	0.9498
	200	0.0035	0.0082	0.0741	0.0829	0.9080	0.9091	0.9529	0.9535
	500	0.0012	0.0036	0.0467	0.0525	0.9030	0.9010	0.9503	0.9492
1.5	10	0.2851	0.2527	0.8907	0.5729	0.9267	0.8892	0.9627	0.9430
	20	0.1094	0.1138	0.4514	0.3319	0.8922	0.8909	0.9446	0.9439
	50	0.0364	0.0437	0.2345	0.1830	0.8990	0.8963	0.9482	0.9467
	100	0.0164	0.0220	0.1595	0.1225	0.8934	0.8956	0.9452	0.9464
	200	0.0076	0.0119	0.1093	0.0852	0.8973	0.8985	0.9473	0.9479
	500	0.0030	0.0047	0.0680	0.0530	0.8981	0.8988	0.9477	0.9481
2.0	10	0.4765	0.2452	1.2648	0.5596	0.9399	0.8888	0.9695	0.9428
	20	0.1984	0.1108	0.6559	0.3220	0.9112	0.8953	0.9546	0.9462
	50	0.0722	0.0429	0.3308	0.1806	0.9005	0.9001	0.9489	0.9487
	100	0.0341	0.0207	0.2169	0.1227	0.8992	0.9011	0.9483	0.9493
	200	0.0187	0.0102	0.1502	0.0850	0.8992	0.8984	0.9483	0.9478
	500	0.0070	0.0040	0.0918	0.0525	0.9034	0.8989	0.9505	0.9481
3.0	10	0.7498	0.2087	1.7994	0.5119	0.9459	0.9074	0.9726	0.9526
	20	0.4066	0.1074	1.1549	0.3228	0.9312	0.8945	0.9650	0.9458
	50	0.1563	0.0444	0.5828	0.1823	0.9104	0.8939	0.9541	0.9455
	100	0.0778	0.0224	0.3743	0.1239	0.9029	0.8999	0.9502	0.9486
	200	0.0416	0.0115	0.2496	0.0850	0.9050	0.8982	0.9513	0.9477
	500	0.0174	0.0048	0.1542	0.0530	0.8990	0.8990	0.9482	0.9482
5.0	10	0.5604	0.0970	1.9143	0.3714	0.9421	0.9440	0.9706	0.9716
	20	0.5651	0.0757	1.6948	0.2778	0.9479	0.9183	0.9736	0.9583
	50	0.3365	0.0415	1.1552	0.1783	0.9240	0.9018	0.9613	0.9497
	100	0.1676	0.0202	0.7654	0.1225	0.9111	0.8968	0.9545	0.9470
	200	0.0792	0.0101	0.5124	0.0847	0.9011	0.9009	0.9493	0.9492
	500	0.0311	0.0037	0.3113	0.0529	0.9022	0.8974	0.9498	0.9473

Table 4: Estimated bias, root mean-squared and coverage probability α and β ($\beta = 2.0$).

		Bias		RMSE		$CP_{90\%}$		$CP_{95\%}$	
α	n	α	β	α	β	α	β	α	β
0.5	10	0.0085	0.3453	0.2402	0.7956	0.8399	0.8807	0.9165	0.9385
	20	0.0022	0.1565	0.1575	0.4514	0.8685	0.8855	0.9319	0.9410
	50	0.0004	0.0569	0.0948	0.2427	0.8908	0.8963	0.9438	0.9467
	100	0.0004	0.0280	0.0667	0.1632	0.8930	0.8983	0.9450	0.9478
	200	0.0004	0.0145	0.0474	0.1129	0.8968	0.8990	0.9470	0.9482
	500	-0.0001	0.0065	0.0297	0.0703	0.8967	0.9045	0.9469	0.9511
1.0	10	0.1012	0.3399	0.4895	0.7702	0.8788	0.8865	0.9374	0.9415
	20	0.0401	0.1505	0.2721	0.4336	0.8879	0.8936	0.9423	0.9453
	50	0.0151	0.0583	0.1575	0.2437	0.8938	0.8965	0.9454	0.9468
	100	0.0075	0.0289	0.1074	0.1630	0.9021	0.9031	0.9498	0.9503
	200	0.0034	0.0145	0.0753	0.1127	0.9001	0.9015	0.9487	0.9495
	500	0.0013	0.0053	0.0475	0.0696	0.9013	0.9017	0.9494	0.9496
1.5	10	0.2876	0.3408	0.8786	0.7782	0.9323	0.8838	0.9656	0.9401
	20	0.1169	0.1585	0.4559	0.4460	0.8911	0.8903	0.9440	0.9436
	50	0.0396	0.0604	0.2367	0.2452	0.8938	0.8946	0.9454	0.9458
	100	0.0199	0.0312	0.1600	0.1651	0.8960	0.8967	0.9466	0.9469
	200	0.0100	0.0140	0.1098	0.1122	0.8946	0.9042	0.9458	0.9509
	500	0.0039	0.0056	0.0677	0.0700	0.9016	0.9009	0.9495	0.9492
2.0	10	0.4538	0.3280	1.2459	0.7562	0.9383	0.8909	0.9687	0.9439
	20	0.2002	0.1549	0.6651	0.4445	0.9123	0.8876	0.9551	0.9421
	50	0.0689	0.0589	0.3319	0.2446	0.9051	0.8939	0.9514	0.9455
	100	0.0356	0.0285	0.2204	0.1640	0.8983	0.9003	0.9478	0.9488
	200	0.0179	0.0127	0.1506	0.1126	0.9002	0.9003	0.9488	0.9488
	500	0.0065	0.0050	0.0922	0.0703	0.8989	0.8992	0.9481	0.9483
3.0	10	0.7609	0.2751	1.8108	0.6710	0.9456	0.9083	0.9724	0.9531
	20	0.4247	0.1507	1.1785	0.4312	0.9321	0.8929	0.9654	0.9450
	50	0.1562	0.0603	0.5821	0.2448	0.9067	0.8949	0.9522	0.9460
	100	0.0699	0.0277	0.3708	0.1627	0.9058	0.8992	0.9517	0.9483
	200	0.0368	0.0132	0.2496	0.1123	0.9034	0.9016	0.9505	0.9495
	500	0.0148	0.0056	0.1536	0.0704	0.9032	0.9020	0.9504	0.9497
5.0	10	0.5371	0.1235	1.9087	0.5012	0.9422	0.9314	0.9707	0.9651
	20	0.5459	0.0960	1.6681	0.3631	0.9468	0.9242	0.9731	0.9614
	50	0.3516	0.0569	1.1766	0.2354	0.9233	0.9019	0.9609	0.9497
	100	0.1844	0.0317	0.7775	0.1635	0.9071	0.8987	0.9524	0.9480
	200	0.0921	0.0160	0.5167	0.1127	0.9035	0.9004	0.9505	0.9489
	500	0.0352	0.0058	0.3129	0.0702	0.9021	0.9036	0.9498	0.9506

Some of the points are very clear from the numerical experiments. Although the biases of $\widehat{\alpha}$ and $\widehat{\beta}$ goes to zero as sample size increase, both parameters are positively biased. It is also seen that the RMSE of both parameters decrease as sample size increase. Interestingly, as the value of α increase their corresponding bias is bigger, while the bias of β decrease. Therefore, estimation of α becomes better for lower values of α whereas the estimation of β are more accurate for large values of α.

Also Tables 1–4 show that, as the sample size increases, the coverage probabilities for α and β are quite close to the nominal levels. Curiously, for large values of α their coverage probability is greater than the nominal levels, while for β the coverage probability is very close.

5 Applications

In this section we present two applications using two published data sets which demonstrate the suitability of the proposed UW distribution. The first data set is from [14] and refer to 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania. The second data set refer to 48 observations obtained from 12 core samples from petroleum reservoirs that were sampled by 4 cross-sections. It should be noted that this data can be found in [26] on a `data.frame` named as `rock`. These data sets are reported in Table 5.

The proposed two-parameter UW distribution is compared with the following two-parameter distributions on the unit interval (0,1)

(i) Beta distribution:

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \qquad \alpha,\beta>0.$$

(ii) Kumaraswamy distribution:

$$f(x;\alpha,\beta) = \alpha\,\beta\, x^{\alpha-1}\,(1-x^{\alpha})^{\beta-1}, \qquad \alpha,\beta>0.$$

(iii) Johnson S_B distribution:

$$f(x;\alpha,\beta) = \frac{\beta}{\sqrt{2\,\pi}}\,\frac{1}{x\,(1-x)}\;\exp\left\{-\frac{1}{2}\left[\alpha+\beta\,\log\left(\frac{x}{1-x}\right)\right]^2\right\}, \qquad \alpha\in\mathbb{R},\beta>0.$$

(iv) Unit-Logistic distribution:

$$f(x;\alpha,\beta) = \frac{\beta\,\mathrm{e}^{\alpha}\,x^{\beta-1}\,(1-x)^{\beta-1}}{\left[x^{\beta}\,\mathrm{e}^{\alpha}+(1-x)^{\beta}\right]^2}, \qquad \alpha\in\mathbb{R},\beta>0.$$

(v) Simplex distribution:

$$f(x;\alpha,\beta) = \left[2\,\pi\,\beta^2\{x\,(1-x)\}^3\right]^{-\frac{1}{2}}\,\exp\left\{-\frac{1}{2\,\beta^2}\left[\frac{(x-\alpha)^2}{x\,(1-x)\,\alpha^2(1-\alpha)^2}\right]\right\}, \qquad \alpha\in(0,1),\beta>0.$$

(vi) Unit-Gamma distribution:

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{\beta-1}\, (-\log x)^{\alpha-1}, \qquad \alpha,\beta > 0.$$

(vii) Extended Arcsine distribution:

$$f(x;\alpha,\beta) = \frac{\alpha\,\beta}{\pi(x-x^2)^{1/2}} \left[1-\frac{2}{\pi}\arcsin\left(\sqrt{x}\right)\right]^{\alpha-1} \left\{1-\left[1-\frac{2}{\pi}\arcsin\left(\sqrt{x}\right)\right]^{\alpha}\right\}^{\beta-1}, \qquad \alpha,\beta > 0.$$

(viii) Exponentiated Topp-Leone distribution:

$$f(x;\alpha,\beta) = 2\,\alpha\,\beta\,(1-x)\,\left[x\,(2-x)\right]^{\alpha-1}\,\left[1-x^{\alpha}\,(2-x)^{\alpha}\right]^{\beta-1}, \qquad \alpha,\beta > 0.$$

Table 5: Flood level data and Petroleum reservoirs data.

Data Set I
0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74
Data Set II
0.09, 0.11, 0.12, 0.12, 0.13, 0.14, 0.15, 0.15, 0.15, 0.15, 0.15, 0.16, 0.16, 0.16, 0.16, 0.17, 0.17, 0.18, 0.18, 0.18, 0.18, 0.19, 0.19, 0.20, 0.20, 0.20, 0.20, 0.20, 0.20, 0.23, 0.23, 0.23, 0.23, 0.24, 0.25, 0.26, 0.26, 0.28, 0.28, 0.28, 0.29, 0.31, 0.33, 0.33, 0.34, 0.42, 0.44, 0.46

The maximum likelihood estimates and their corresponding standard errors for both data sets are given in Table 6. To check the suitability of the suitability of the UW distribution and the above eight competing distributions, we consider three goodness-of-fit tests (Kolmogorov-Smirnov statistic (KS), Anderson-Darling statistic (AD) and Cramér-von Mises statistic (CvM)). In order to compare the UW distributions to the above eight competing distributions, we consider the likelihood-based statistics (Akaike's Information Criterion (AIC) and the Bayesian information criterion (BIC)). The results for both data sets are presented in Table 7.

Table 6: Maximum likelihood estimate (standard-error) for α and β.

	Data Set I		Data Set II	
Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$
UW	1.0248	3.9036	0.0602	5.1130
	(0.2399)	(0.6806)	(0.0236)	(0.5754)
i	6.7569	9.1117	5.9422	21.2070
	(2.0946)	(2.8518)	(1.1815)	(4.3472)
ii	3.3634	11.7906	2.7186	44.6540
	(0.6034)	(5.3604)	(0.2935)	(17.5720)
iii	0.6143	1.9262	2.8736	2.1525
	(0.2438)	(0.3045)	(0.3269)	(0.2197)
iv	1.3599	3.5915	5.2285	3.8274
	(0.4793)	(0.6886)	(0.6913)	(0.4611)
v	0.4309	1.0923	0.2197	1.1637
	(0.0269)	(0.1727)	(0.0113)	(0.1188)
vi	8.7310	9.7251	17.9510	11.3100
	(2.7099)	(3.1068)	(3.6307)	(2.3197)
vii	9.1631	141.9528	14.1764	101.1463
	(1.7151)	(126.7324)	(1.6670)	(52.7009)
viii	4.6064	4.0442	3.1358	13.6413
	(0.9496)	(1.4752)	(0.3642)	(4.1988)

UW: unit-Weibull, i: Beta, ii: Kumaraswamy, iii: Johnson S_B, iv: Unit-Logistic, v: Simplex, vi: Unit-Gamma, vii: Extended Arcsine and viii: Exponentiated Topp-Leone.

A close inspection of Table 7 reveals that the UW distribution outperforms the competing distributions for both data sets, since it has the smallest AIC and BIC values. This conclusion is also support by the probability-probability plots in Figures 4 and 5, where we can see again that the UW provides the best fit among the considered models.

Table 7: Goodness-of-fit measures (p-values) and likelihood-based statistics.

	Data Set I				
Distribution	KS	CvM	AD	AIC	BIC
UW	0.1448 (0.7958)	0.0512 (0.8742)	0.3434 (0.9013)	-28.3430	-26.3515
i	0.1988 (0.4082)	0.1236 (0.4847)	0.7327 (0.5303)	-24.1245	-22.1330
ii	0.2109 (0.3359)	0.1636 (0.3528)	0.9322 (0.3936)	-21.7324	-19.7409
iii	0.1935 (0.4424)	0.1153 (0.5187)	0.6930 (0.5627)	-24.5257	-22.5342
iv	0.1391 (0.8339)	0.0547 (0.8529)	0.4804 (0.7648)	-25.4724	-23.4809
v	0.2098 (0.3424)	0.1447 (0.4087)	0.7970 (0.4815)	-24.3065	-22.3150
vi	0.1955 (0.4293)	0.1178 (0.5084)	0.7046 (0.5530)	-24.3769	-22.3854
vii	0.1543 (0.7275)	0.0564 (0.8419)	0.3890 (0.8581)	-27.8320	-25.8405
viii	0.2063 (0.3625)	0.1432 (0.4136)	0.8142 (0.4692)	-23.1852	-21.1937
	Data Set II				
Distribution	KS	CvM	AD	AIC	BIC
UW	0.1007 (0.7143)	0.0383 (0.9434)	0.2338 (0.9782)	-112.2416	-108.4992
i	0.1428 (0.2819)	0.1301 (0.4577)	0.7771 (0.4971)	-107.2004	-103.4580
ii	0.1533 (0.2092)	0.2060 (0.2566)	1.2892 (0.2358)	-100.9831	-97.2407
iii	0.1252 (0.4390)	0.0862 (0.6587)	0.5190 (0.7267)	-109.9699	-106.2275
iv	0.0979 (0.7467)	0.0557 (0.8435)	0.4054 (0.8427)	-109.9063	-106.1639
v	0.1297 (0.3945)	0.0965 (0.6041)	0.5569 (0.6888)	-110.1133	-106.3709
vi	0.1365 (0.3325)	0.1130 (0.5263)	0.6793 (0.5756)	-108.2175	-104.4751
vii	0.1138 (0.5628)	0.0492 (0.8829)	0.2957 (0.9411)	-111.9385	-108.1961
viii	0.1525 (0.2145)	0.1866 (0.2957)	1.1477 (0.2880)	-102.7118	-98.9694

UW: Unit-Weibull, i: Beta, ii: Kumaraswamy, iii: Johnson S_B, iv: Unit-Logistic, v: Simplex, vi: Unit-Gamma, vii: Extended Arcsine and viii: Exponentiated Topp-Leone.

In order to discriminate between the UW distribution with each competing distribution, we apply the Vuong likelihood ratio test of non-nested distributions [34]. The Vuong test statistic is defined as

$$T = \frac{1}{\widehat{\omega}^2 \sqrt{n}} \sum_{i=1}^{n} \log \frac{f(x_i \mid \boldsymbol{m}\widehat{\theta})}{g(x_i \mid \boldsymbol{m}\widehat{\gamma})}$$

where

$$\widehat{\omega}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log\frac{f(x_i \mid \boldsymbol{m}\widehat{\theta})}{g(x_i \mid \boldsymbol{m}\widehat{\gamma})}\right)^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\left(\log\frac{f(x_i \mid \boldsymbol{m}\widehat{\theta})}{g(x_i \mid \boldsymbol{m}\widehat{\gamma})}\right)\right]^2$$

is an estimator for the variance of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\log\frac{f(x_i \mid \boldsymbol{m}\widehat{\theta})}{g(x_i \mid \boldsymbol{m}\widehat{\gamma})}$, $f(x_i \mid \boldsymbol{m}\widehat{\theta})$ and $g(x_i \mid \boldsymbol{m}\widehat{\gamma})$ are the corresponding rival densities evaluated at the maximum likelihood estimates. It was demonstrated that, when $n \to \infty$, $T \xrightarrow{D} N(0,1)$. At a significance level $\gamma\%$, we reject distribution equivalence if $|T| < z_{\gamma/2}$, where z_q is the upper q-th quantile of the standard normal distribution.

The results of Voung test are given in Table 8. This table shows that UW and Extended Arcsine distributions are equivalent for data set I. This table also shows that UW and Beta, Johnson S_B, Unit-Logistic, Simplex, Unit-Gamma and Extended Arcsine distributions are equivalent for data set II. However, the UW distribution has smallest AIC and BIC as shown in Table 7. Therefore the UW distribution provides the best fit among the competing distributions.

Table 8: Observed values of Voung statistic (p-values).

Comparisons	Data Set I	Data Set II
UW vs Beta	2.4722 (0.0067)	1.4354 (0.0756)
UW vs Kumarasawamy	2.7266 (0.0032)	2.2934 (0.0109)
UW vs Johnson S_B	2.3338 (0.0098)	0.9908 (0.1609)
UW vs Unit-Logistic	2.3340 (0.0098)	1.0926 (0.1373)
UW vs Simplex	2.0095 (0.0222)	0.9470 (0.1718)
UW vs Unit-Gamma	2.4346 (0.0075)	1.2882 (0.0988)
UW vs Extended Arcsine	1.4977 (0.0671)	0.2632 (0.3962)
UW vs Exponentiated Topp-Leone	2.5305 (0.0057)	2.0813 (0.0187)

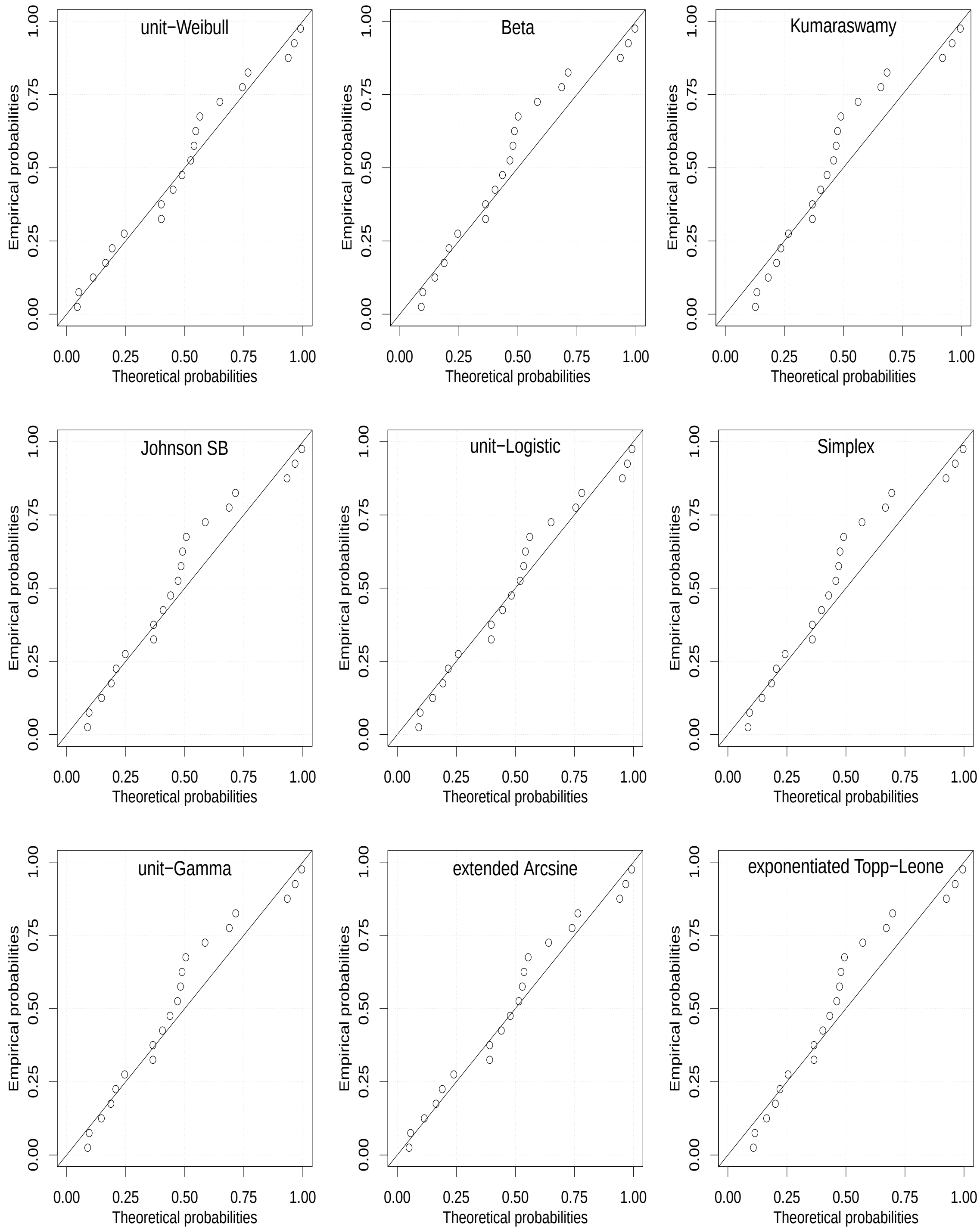

Figure 4: PP-Plots of the fitted distributions — Data Set I.

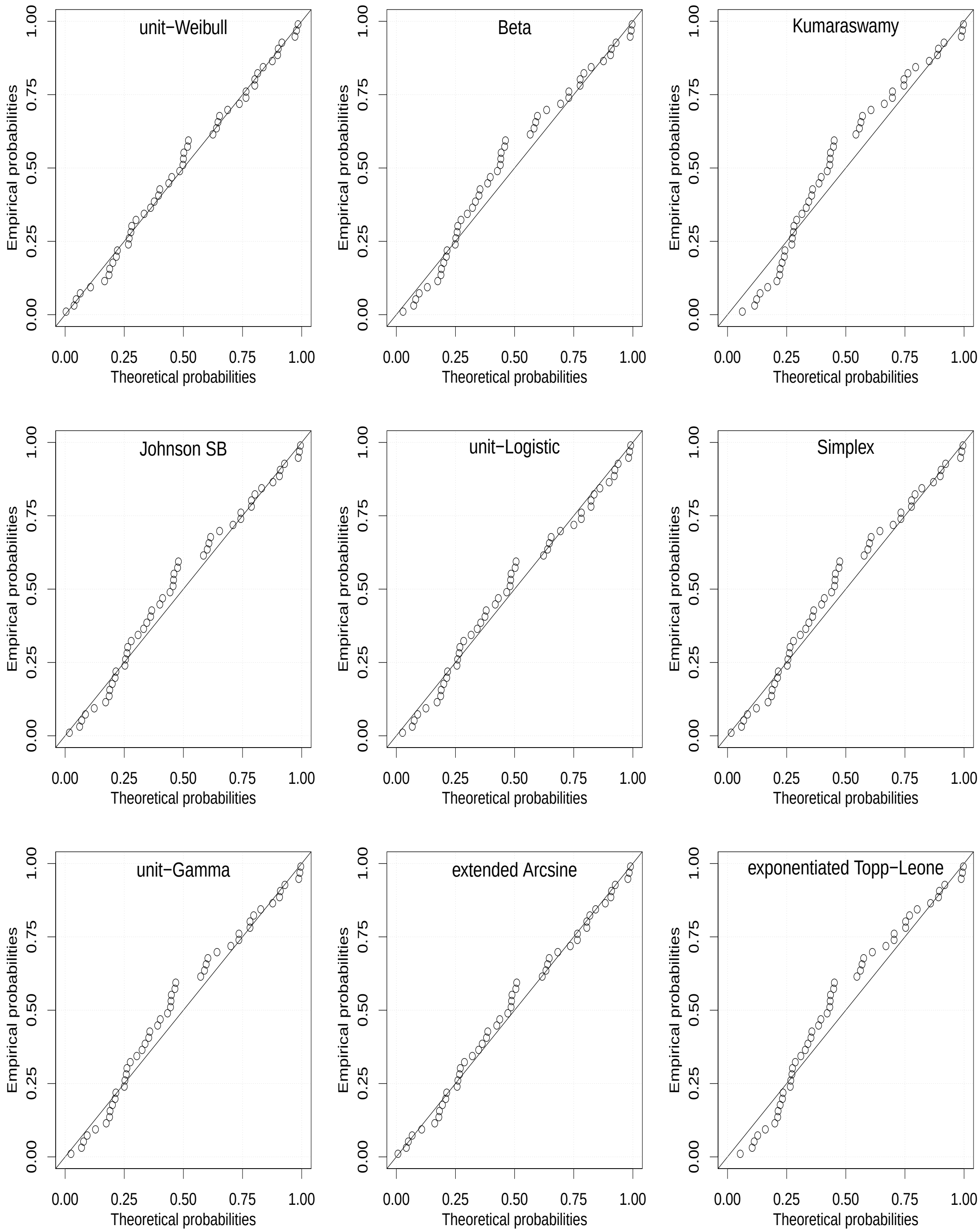

Figure 5: PP-Plots of the fitted distributions — Data Set II.

6 Conclusions

In this paper a new two-parameter distribution, called the UW distribution with support on $(0, 1)$, is introduced and studied in details. Maximum likelihood estimators of the parameters and their standard errors were derived. We also proposed a starting-point strategy using the fact that UW cumulative distribution function can be linearized. Random sample for the distribution can be easily simulated by simple transformation of samples generated from the Weibull distribution. A simulation study was carried out to examine the bias and root mean-squared error of the maximum likelihood estimators of the parameters as well as the coverage probability of the confidence intervals. Applications of the proposed distribution to two real data sets showed better fit than many other well-known two-parameter distributions with support on (0,1), such as Beta, Kumaraswamy, Johnson S_B, unit-Logistic, Simplex, unit-Gamma, extended Arcsine, and exponentiated Topp-Leone distributions. Finally, since the UW is very flexible and has a closed form expression for the quantiles, its quantile regression model will be useful in many applications and is now under investigation.

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