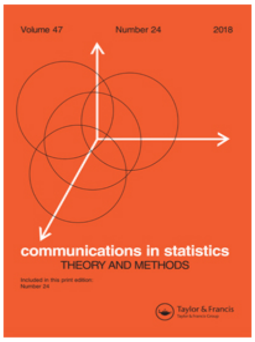

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The unit-inverse Gaussian distribution: A new alternative to two-parameter distributions on the unit interval

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ABSTRACT
A new two-parameter distribution over the unit interval, called the Unit-Inverse Gaussian distribution, is introduced and studied in detail. The proposed distribution shares many properties with other known distributions on the unit interval, such as Beta, Johnson S_B, Unit-Gamma, and Kumaraswamy distributions. Estimation of the parameters of the proposed distribution are obtained by transforming the data to the inverse Gaussian distribution. Unlike most distributions on the unit interval, the maximum likelihood or method of moments estimators of the parameters of the proposed distribution are expressed in simple closed forms which do not need iterative methods to compute. Application of the proposed distribution to a real data set shows better fit than many known two-parameter distributions on the unit interval.

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Beta distribution; equilibrium distribution; inverse Gaussian distribution; length biased distribution; maximum likelihood estimation; method of moments estimation; weighted distribution

1. Introduction

In dealing with the uncertainty of a bounded phenomena, e.g. proportion of a certain characteristic, a continuous distribution with a bounded domain is needed to describe such phenomena. The only continuous bounded distribution discussed extensively in the literature is the well-known Beta distribution (or the Pearson type IV distribution), whose origin can be traced to 1676 in a letter from Sir Isaac Newton to Henry Oldenberg.

Over the years, many continuous distributions with bounded domains were introduced and applied to model uncertainty of a bounded phenomena in different applied fields. For example, Johnson S_B distribution (Johnson 1949), Johnson S_B' distribution (Johnson 1955), Unit-Logistic distribution (Tadikamalla, and Johnson 1982), Topp-Leone distribution (Topp and Leone 1955), Unit-Gamma distribution (Consul and Jain 1971; Grassia 1977; Tadikamalla 1981; Mazucheli, Menezes, and Dey 2017), the Kumaraswamy distribution (Kumaraswamy 1980), the Arcsine distribution (Arnold and Groeneveld 1980), the McDonald's Generalized Beta type I distribution (McDonald 1984), the Simplex distribution (Barndorff-Nielsen and Jørgensen 1991), the Reflected Generalized Topp-Leone distribution (van Drop and Kotz 2006), the Beta Power distribution (Cordeiro and dos Santos 2012), the McDonald Arcsine distribution (Cordeiro and Lemonte 2014), the log-Lindley distribution (Gómez-Déniz, Sordo, and Calderín-Ojeda 2013), the Exponentiated Kumaraswamy distribution (Lemonte, Barreto-Souza, and Cordeiro 2013), the Exponentiated Topp-Leone distribution (Pourdarvish, Mirmostafaee, and Naderi 2015), the Marshall–Olkin Extended

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Kumaraswamy (Castellares and Lemonte 2015), the Reflected Generalized Topp-Leone Power series distribution (Condino and Domma 2016), the Transmuted Kumaraswamy distribution (Shuaib, Robert, and Lena 2016), the size biased Kumaraswamy distribution (Sharma, and Chakrabarty 2016) and the Extended Arcsine distribution (Cordeiro, Lemonte, and Campelo 2016).

It should be pointed out that the majority of these distributions have more than two parameters and their ranges depend on some of the parameters, which makes estimation problematic.

Some of the finite range distributions in the literature are derived from standard distributions by mathematical transformation. For example, the following transformations give rise to distributions on the unit interval:

(i) $X = 1 - \frac{1}{1+\exp[(Z-\alpha)/\beta]}$, where $Z \sim N(0,1)$, $Z \sim$ Laplace$(0,1)$, and $Z \sim$ Logistic$(0,1)$, implies $X \sim$ Johnson $S_B(\alpha,\beta)$, $X \sim$ Johnson $S'_B(\alpha,\beta)$, and $X \sim$ Unit-logistic(α,β), respectively, where $\alpha \in \mathbb{R}, \beta > 0$.

(ii) $X = e^{-Y}$, where $Y \sim$ Gamma(α,β), $Y \sim$ Exponentiated-Exponential(α,β), and $Y \sim$ Lindley(α,β), implies $X \sim$ Unit-Gamma(α,β), $X \sim$ Kumaraswamy(α,β), and $X \sim$ Log-Lindley(α,β), respectively, where $\alpha,\beta > 0$.

Recently, Jódra and Jiménez-Gamero (2016) explored the problematic issues of estimation of the log-Lindley distribution using the method of moments and maximum likelihood.

The aim of this paper is to propose a new distribution on the unit interval, called the Unit-Inverse-Gaussian distribution and discuss some of its properties. These include the shapes of the density and hazard rate functions as well as the moments and associated measures. Maximum likelihood estimation of the model parameters and their asymptotic standard errors are derived. Application of the model to a real data set is finally presented and compared to the fit attained by some other well-known two-parameter distributions on the unit interval, such as Johnson S_B, Beta, Unit-Gamma and Kumaraswamy.

2. Probability density function

A random variable Y is said to have the inverse-Gaussian distribution with parameters μ and λ if its p.d.f. is given by

$$f_Y(y) = \sqrt{\frac{\lambda}{2\pi}}\,\frac{1}{y^{3/2}}\,\exp\left[-\frac{\lambda}{2\mu^2 y}(y-\mu)^2\right], \qquad y > 0, \quad \mu, \lambda > 0 \tag{1}$$

where μ is the mean and λ is a scale parameter.

The cumulative distribution function (c.d.f.) of Y can be written as

$$F_Y(y) = \Phi\left(\sqrt{\frac{\lambda}{y}}\left(\frac{y}{\mu}-1\right)\right) + e^{2\lambda/\mu}\,\Phi\left(-\sqrt{\frac{\lambda}{y}}\left(\frac{y}{\mu}+1\right)\right) \tag{2}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution. Also, the moment generating function (m.g.f.) of Y is given by

$$M_Y(t) = \exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 t}{\lambda}}\right)\right], \qquad -\infty < t < \frac{\lambda}{2\mu^2} \tag{3}$$

A complete guide to the Inverse-Gaussian distribution can be found in Seshadri (1999).

Now, consider the transformation $X = e^{-Y}$. Then X is said to have the Unit-Inverse-Gaussian (UIG) distribution with p.d.f.

$$f(x) = \sqrt{\frac{\lambda}{2\pi}} \frac{1}{x(-\log x)^{3/2}} \exp\left[\frac{\lambda}{2\mu^2 \log x}(\log x + \mu)^2\right], \quad 0 < x < 1 \tag{4}$$

and

$$f(0) = \begin{cases} \infty & \text{if } 0 < \frac{\lambda}{2\mu^2} < 1, \\ 0 & \text{if } \frac{\lambda}{2\mu^2} \geq 1, \end{cases} \qquad f(1) = 0$$

Because of the special transformation $X = e^{-Y}$, many properties of the UIG random variable X can be easily derived using properties of the IG random variable Y. For example,

(i) If $X \sim UIG(\mu, \lambda)$, then $E(X^r) = M_Y(-r)$ for any $r = 1, 2, \ldots$.
(ii) If $X \sim UIG(\mu, \lambda)$, then $\overline{F}(x) = P(X > x) = F_Y(-\log x)$.
(iii) If $X \sim UIG(\mu, \lambda)$, then $X^c \sim UIG(c\mu, c\lambda)$ for any $c > 0$.
(iv) If $X_i \sim UIG(\mu, \lambda)$, $1 \leq i \leq n$, are independent and identically distributed, then $\prod_{i=1}^n X_i \sim UIG(n\mu, n^2\lambda)$.
(v) If $X_i \sim UIG(\mu_i, 2\mu_i^2)$, $1 \leq i \leq n$, are independent, then $\prod_{i=1}^n X_i \sim UIG(\sum_{i=1}^n \mu_i, 2 (\sum_{i=1}^n \mu_i)^2)$.
(vi) Suppose $Y|\lambda \sim IG(\mu, \lambda)$ where $\lambda \sim Gamma(\alpha, \beta)$ with p.d.f. $g(\lambda)$.
The unconditional p.d.f. of Y is given by

$$\begin{aligned} f_Y^*(y) &= \int_0^\infty f_Y(y|\lambda)\,.\,g(\lambda)\,d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \frac{1}{y^{3/2}} \int_0^\infty \lambda^{\alpha-\frac{1}{2}} \exp\left\{-\left[\beta + \frac{1}{2\mu^2 y}(y-\mu)^2\right]\lambda\right\} d\lambda \\ &= \frac{\Gamma(\alpha+\frac{1}{2})\,\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \frac{1}{y^{3/2}} \frac{1}{\left[\beta + \frac{1}{2\mu^2 y}(y-\mu)^2\right]^{\alpha+\frac{1}{2}}}, \qquad y > 0 \end{aligned}$$

Now, the p.d.f. of $X = e^{-Y}$ is given by

$$f^*(x) = \frac{\Gamma(\alpha+\frac{1}{2})\,\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \frac{1}{x(-\log x)^{3/2}} \frac{1}{\left[\beta - \frac{1}{2\mu^2 \log x}(\log x + \mu)^2\right]^{\alpha+\frac{1}{2}}}, \quad 0 < x < 1$$

In particular, when $\alpha = 1$, we have

$$f^*(x) = \frac{\beta\mu^3}{x\left[-2\beta\mu^2 \log x + (\log x + \mu)^2\right]^{3/2}}, \qquad 0 < x < 1$$

(vii) The weighted UIG distribution with p.d.f.

$$f_s(x) = \frac{x^s}{E(X^s)} f(x), \qquad 0 < x < 1, \quad -\infty < s < \infty$$

where

$$E(X^s) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2s\mu^2}{\lambda}}\right)\right], \qquad \lambda + 2s\,\mu^2 > 0$$

is the UIG distribution with reparametrisation

$$\mu \to \mu_s = \mu \sqrt{\frac{\lambda}{\lambda + 2s\,\mu^2}}, \qquad \lambda + 2s\,\mu^2 > 0$$

When $s = 1$, we obtain the length-biased UIG distribution.

On the other hand, some properties of the UIG can be derived without using properties of the IG distribution. For example,

(i) the p.d.f. of the smallest order statistic $Y_1 = \min(X_1, \dots, X_n)$ from the UIG distribution is given by

$$g_1(y_1) = n\left[1 - F(y_1)\right]^{n-1} f(y_1), \qquad 0 < y_1 < 1$$

(ii) the p.d.f. of the largest order statistic $Y_n = \max(X_1, \dots, X_n)$ from the UIG distribution is given by

$$g_n(y_n) = n\left[F(y_n)\right]^{n-1} f(y_n), \qquad 0 < y_n < 1$$

The following theorem shows all possible shapes of the p.d.f. of the UIG distribution.

Theorem 1. *Let $c = \frac{\lambda}{2\mu^2}$ and $\Delta = 9 - 8(1-c)\lambda$. The p.d.f. $f(x)$ of the UIG distribution is*

(i) decreasing if $0 < c < 1$ and $\Delta \le 0$,

(ii) decreasing-increasing-decreasing if $0 < c < 1$ and $\Delta > 0$,

(iii) unimodal if $c \ge 1$.

Proof. The first derivative of $f(x)$ is given by

$$f'(x) = -\frac{f(x)}{2x(\log x)^2}\,\xi(\log x) \tag{5}$$

where

$$\xi(z) = 2(1-c)z^2 + 3z + \lambda, \qquad -\infty < z = \log(x) < 0$$

with

$$\xi(-\infty) = \begin{cases} \infty & \text{if } 0 < c < 1, \\ -\infty & \text{if } c \ge 1, \end{cases} \qquad \xi(0) = \lambda$$

Let $\Delta = 9 - 8(1-c)\lambda$ be the discriminant of the quadratic equation $\xi(z) = 0$. Now

(i) if $0 < c < 1$ and $\Delta \le 0$, the function $\xi(z)$ is non-negative on $(-\infty, 0)$,

(ii) if $0 < c < 1$ and $\Delta > 0$, the function $\xi(z)$ has two real zeros on $(-\infty, 0)$ and changes sign from positive to negative to positive,

(iii) if $c \ge 1$, i.e. $\Delta > 0$, the function $\xi(z)$ has a single real zero on $(-\infty, 0)$ and changes sign from negative to positive.

Since the sign of $f'(\cdot)$ is opposite to the sign of $\xi(\cdot)$, the theorem follows. □

The conditions of Theorem 1 are inequalities on the parameters λ and μ. These inequalities can be solved to give the following simplified conditions:

(i) $\lambda > \frac{9}{8}$ and $\mu \ge \frac{2\lambda}{\sqrt{8\lambda - 9}}$.

(ii) $0 < \lambda \le \frac{9}{8}$ and $\mu > \sqrt{\frac{\lambda}{2}}$ or $\lambda > \frac{9}{8}$ and $\sqrt{\frac{\lambda}{2}} < \mu < \frac{2\lambda}{\sqrt{8\lambda - 9}}$.

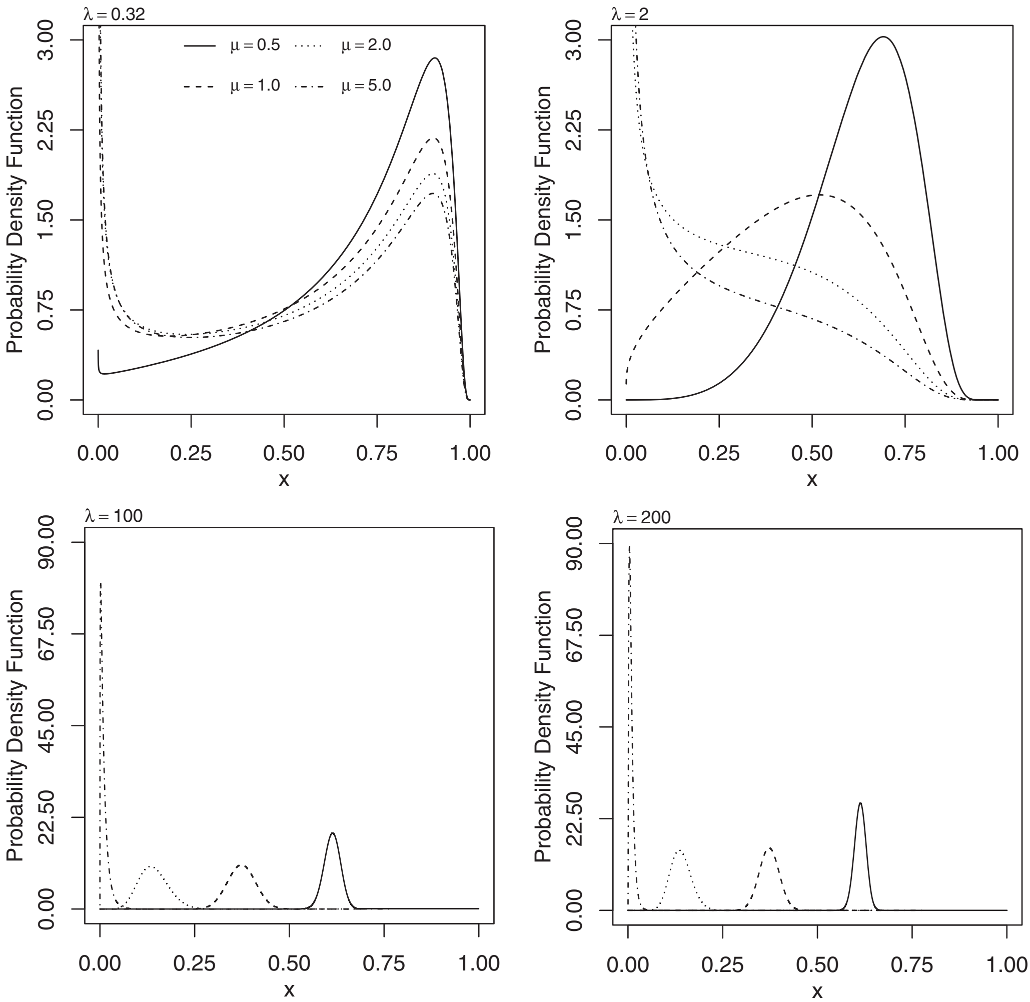

Figure 1. Probability density function of UIG distribution for selected values of μ and λ.

(iii) $\lambda > 0$ and $\mu \le \sqrt{\frac{\lambda}{2}}$.

Figure 1 shows the three possible shapes of the p.d.f. of the UIG distribution. The unimodal and decreasing shapes are also possessed by the Beta distribution while the decreasing-increasing-decreasing shape is not. On the other hand, of course, the Beta distribution also includes anti-unimodal and increasing density shapes.

3. Hazard rate function

Since the survival function (s.f.) of the UIG distribution is given by

$$S(x) = P(X > x) = F_Y(-\log x)$$
$$= \Phi\left(\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu} - 1\right)\right) + e^{2\lambda/\mu}\,\Phi\left(-\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu} + 1\right)\right) \quad (6)$$

it follows that the hazard rate function (h.r.f.) of the UIG distribution is given by

$$h(x) = \frac{f(x)}{S(x)}$$
$$= \frac{\sqrt{\frac{\lambda}{2\pi}}\,\frac{1}{x\,(-\log x)^{3/2}}\,\exp\left[\frac{\lambda}{2\mu^2\log x}(\log x+\mu)^2\right]}{\Phi\left(\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu}-1\right)\right)+e^{2\lambda/\mu}\,\Phi\left(-\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu}+1\right)\right)}, \quad 0<x<1 \tag{7}$$

To determine the shapes of the h.r.f. $h(x)$, we use the results of Ghitany (2004) and Gupta and Warren (2001). These results are sufficient conditions in terms of the function $\eta(x) = -f'(x)/f(x)$. In particular, the following results will be used in the proof of Theorem 2 below.

(R1) If $\eta(x)$ is increasing, then $h(x)$ is increasing (Ghitany 2004).

(R2) If $\eta(x)$ is decreasing-increasing (anti-unimodal), $f(0) = \infty$ and $f(1) = 0$, then $h(x)$ is anti-unimodal (Ghitany 2004).

(R3) Suppose $\eta(x)$ is increasing-decreasing-increasing and $\eta'(x) = 0$ has two zeros $x_1 < x_2$. If $h(x)$ increases in a neighbourhood of zero and $h'(x_1) \geq 0$ ($h'(x_1) < 0$), then $h(x)$ is increasing (increasing-decreasing-increasing) (Gupta and Warren 2001).

(R4) Suppose $\eta'(x) = 0$ has three zeros $x_1 < x_2 < x_3$. The h.r.f. $h(x)$ has at most three critical points. Specifically, $h(x)$ has at most one critical point in each of the intervals $[0, x_1]$, $[x_1, x_2]$ and $[x_2, x_3]$ (Gupta and Lvin 2005).

Note that $h(0) = f(0)$ and $h(1) = \eta(1)$. Also, $h'(x) = h(x)[h(x) - \eta(x)]$.

The following theorem shows all possible shapes of the h.r.f. of the UIG distribution.

Theorem 2. *Let* $c = \frac{\lambda}{2\mu^2}$,

$$\phi(z) = 2(1-c)z^3 + 3z^2 + (\lambda+3)z + 2\lambda, \qquad -\infty < z < 0$$

$$z_0 = \begin{cases} -\frac{\lambda+3}{6} & \text{if } c = 1 \\ \frac{1}{2(c-1)}\left[1-\sqrt{1+\frac{2}{3}(c-1)(\lambda+3)}\right] & \text{if } c > 1 \end{cases}$$

and z_1 *is the smallest zero of* $\phi(z)$. *The h.r.f.* $h(x)$ *of the UIG distribution is*

(i) anti-unimodal if $0 < c < 1$,

(ii) increasing if (a) $c \geq 1$ *and* $\phi(z_0) \geq 0$ *or (b)* $c \geq 1$, $\phi(z_0) < 0$ *and* $h'(e^{z_1}) \geq 0$,

(iii) increasing-decreasing-increasing if $c \geq 1$, $\phi(z_0) < 0$ *and* $h'(e^{z_1}) < 0$.

Proof. From Equation (5), we have

$$\eta(x) = -\frac{f'(x)}{f(x)} = \frac{1}{2x}\left[2(1-c) + \frac{3}{\log x} + \frac{\lambda}{(\log x)^2}\right] \tag{8}$$

with

$$\eta(0) = \begin{cases} \infty & \text{if } 0 < c < 1, \\ -\infty & \text{if } c \geq 1, \end{cases} \qquad \eta(1) = \infty$$

Therefore,

$$\eta'(x) = \frac{-1}{2x^2\,(\log x)^3}\,\phi(\log x)$$

where

$$\phi(z) = 2(1-c)z^3 + 3z^2 + (3+\lambda)z + 2\lambda, \qquad -\infty < z = \log(x) < 0$$

with

$$\phi(-\infty) = \begin{cases} -\infty & \text{if } 0 < c < 1, \\ \infty & \text{if } c \geq 1, \end{cases} \qquad \phi(0) = 2\lambda$$

Clearly, the sign of $\eta'(\cdot)$ is the same sign of $\phi(\cdot)$.

(i) For $0 < c < 1$, let $D = 1 - \frac{2}{3}(1-c)(\lambda+3)$ and

$$z_1^* = \frac{-1-\sqrt{D}}{2(1-c)}, \qquad z_2^* = \frac{-1+\sqrt{D}}{2(1-c)}$$

are the critical points of $\phi(z)$ where $\phi(z_1^*) > 0$ and $\phi(z_2^*) < 0$.

If (a) $D \leq 0$ or (b)$D > 0, \phi(z_1^*) \leq 0$ or (c) $D > 0, \phi(z_2^*) \geq 0$, then $\phi(z)$ has a unique zero and changes sign from negative to positive. Therefore, $\eta'(x)$ also has a unique zero and changes sign from negative to positive, i.e. $\eta(x)$ is anti-unimodal.

Since $f(0) = \infty$ and $f(1) = 0$, the h.r.f. $h(x)$ is also anti-unimodal, see result (R2) above.

Now, we investigate the shape of the h.r.f. $h(x)$ when (d) $D > 0$, $\phi(z_1^*) > 0, \phi(z_2^*) < 0$. In this case, $\phi(z)$ has three zeros $z_{01} < z_{02} < z_{03}$. Therefore, $\eta'(x) = 0$ has also three zeros $x_1 = e^{z_{01}} < x_2 = e^{z_{02}} < x_3 = e^{z_{03}}$, i.e. $x_1 < x_2 < x_3$ are the three critical points of $\eta(x)$.

Since $\Delta = 9[1 - \frac{8}{9}(1-c)\lambda] > 9[1 - \frac{2}{3}(1-c)(\lambda+3)] = 9D > 0$, the function $\eta(x)$ has exactly two zeros $x_{01} = e^{y_1} < x_{02} = e^{y_2}$ where

$$y_1 = -\frac{3+\sqrt{\Delta}}{4(1-c)}, \qquad y_2 = -\frac{3-\sqrt{\Delta}}{4(1-c)}$$

Note that

$$y_1 < -\frac{3+\sqrt{9D}}{4(1-c)} < -\frac{3+\sqrt{9D}}{6(1-c)} = z_1^*, \qquad y_2 > -\frac{3-\sqrt{9D}}{4(1-c)} > -\frac{3-\sqrt{9D}}{6(1-c)} = z_2^*$$

and

$$\phi(y_1) = -\frac{\Delta + 3\sqrt{\Delta}}{4(1-c)} < 0, \qquad \phi(y_2) = \frac{2\lambda\sqrt{\Delta}}{3+\sqrt{\Delta}} > 0$$

Since $\phi(z)$ is increasing on $(-\infty, z_1^*)$ with $\phi(z_{01}) = 0$, we have $y_1 < z_{01} < z_1^*$. Similarly, since $\phi(z)$ is increasing on $(z_2^*, 0)$ with $\phi(z_{03}) = 0$, we have $z_2^* < z_{03} < y_2$.

It follows that $x_{01} = e^{y_1} < x_1 = e^{z_{01}}$ where $\eta(x_{01}) = 0$ and $\eta(x_1) < 0$. Similarly, we have $x_3 = e^{z_{03}} < x_{02} = e^{y_2}$ where $\eta(x_3) < 0$ and $\eta(x_{02}) = 0$.

Since $\eta(0) = \infty$, $\eta(x_{01}) = 0$, $\eta(x_1) < 0$, $\eta(x_3) < 0$, $\eta(x_{02}) = 0$, and $\eta(1) = \infty$, we also have $\eta(x_2) < 0$. In summary, $\eta(x) < 0$ for all $x \in (x_{01}, x_{02})$.

Now, since $h'(x) = h(x)[h(x) - \eta(x)] > 0$ for all $x \in [x_1, x_3]$, $h(x)$ is increasing on each of the intervals $[x_1, x_2]$ and $[x_2, x_3]$, i.e. $h(x)$ has no critical points on these intervals, see result (R4) above.

Finally, since $h(0) = h(1) = \infty$ for $0 < c < 1$, it follows that $h(x)$ has an absolute minimum on $[0, x_1]$, i.e. $h(x)$ is anti-unimodal.

(ii) If $c \geq 1$, the function $\phi(z)$ has an absolute minimum at the point z_0 given by

$$z_0 = \begin{cases} -\frac{\lambda+3}{6} & \text{if } c = 1, \\ \frac{1}{2(c-1)}\left[1 - \sqrt{1 + \frac{2}{3}(c-1)(\lambda+3)}\right] & \text{if } c > 1. \end{cases}$$

(iia) If $\phi(z_0) \geq 0$, then $\phi(z)$, and hence $\eta'(x)$, is non negative. That is, $\eta(x)$ is increasing implying that $h(x)$ is increasing, see result (R1) above.

(iib and iii) If $\phi(z_0) < 0$, then the equation $\phi(z) = 0$ has two roots $z_1 < z_2$ on $(-\infty, 0)$. Since, in this case, $\phi(-\infty) = \infty$ and $\phi(0) = 2\lambda$, it follows that $\phi(z)$, and hence $\eta'(x)$, changes sign from positive to negative to positive. That is, $\eta(x)$ is increasing-decreasing-increasing with critical points $x_1 = e^{z_1} < x_2 = e^{z_2}$.

Since, in this case, $h(x)$ increases in a neighbourhood of zero, it follows that $h(x)$ is increasing (increasing-decreasing-increasing) if $h'(x_1) \geq 0$ ($h'(x_1) < 0$), see result (R3) above. □

For Theorem 2, condition (i) can be replaced by the condition:

$$\lambda > 0 \text{ and } \mu > \sqrt{\frac{\lambda}{2}}$$

Also, condition (ii) when $c = 1$ can be replaced by the condition:

$$9 - 6\sqrt{2} < \lambda < 9 + 6\sqrt{2} \text{ and } \mu = \sqrt{\frac{\lambda}{2}}$$

The remaining conditions of Theorem 2 cannot be expressed in simple conditions on the parameters λ and μ because of the implicit nature of the smallest root z_1 and the complex nature of the derivative $h'(e^{z_1})$ of the hazard rate function $h(\cdot)$.

Figure 2 shows the possible shapes of the h.r.f. of the UIG distribution. The anti-unimodal and increasing shapes are also possessed by the Beta distribution (see Ghitany (2004)) while the increasing-decreasing-increasing shape is not.

Remark.

(i) The cases $\{\lambda = 0.32, \mu = 1, 2, 5\}$ and $\{\lambda = 2, \mu = 2, 5\}$ in Figure 2 imply that $0 < c < 1$, $D < 0$, $\phi(z)$ has a unique zero, and $\eta(x)$ has also a unique critical point.

(ii) The case $\{\lambda = 0.32, \mu = 0.5\}$ in Figure 2 implies that $c = 0.64$, $D = 0.203$, $z_1^* = -2.015$, $z_2^* = -0.763$, $\phi(z_1^*) = 0.240$, $\phi(z_2^*) = -0.466$. In this case, $\phi(z)$ has three zeros $z_{01} = -2.399$, $z_{02} = -1.525$, $z_{03} = -0.243$. Moreover, $\eta(x)$ has also three critical points $x_1 = 0.091$, $x_2 = 0.218$, $x_3 = 0.784$ with $\eta(x_1) = -2.615$, $\eta(x_2) = -2.549$, $\eta(x_3) = -3.957$.

The UIG distribution on the unit interval is naturally suitable for modeling rates and proportions. An extended version of the UIG distribution using the transformation $T = \nu X$, where ν is known positive value, can be useful for modeling continuous distributions with bounded support $(0, \nu)$. For example, T represents the failure time (in days) of a device during a warranty period of $\nu = 365$ days, or the time (in minutes) of the first kick goal scored by a team in a professional soccer match of $\nu = 90$ minutes (see Meintanis (2007)).

4. Moments and associated measures

The rth raw moment of the UIG distribution is given by

$$\mu'_r = E(X^r) = E(e^{-rY}) = M_Y(-r) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2r\mu^2}{\lambda}}\right)\right], \qquad r = 1, 2, \ldots$$

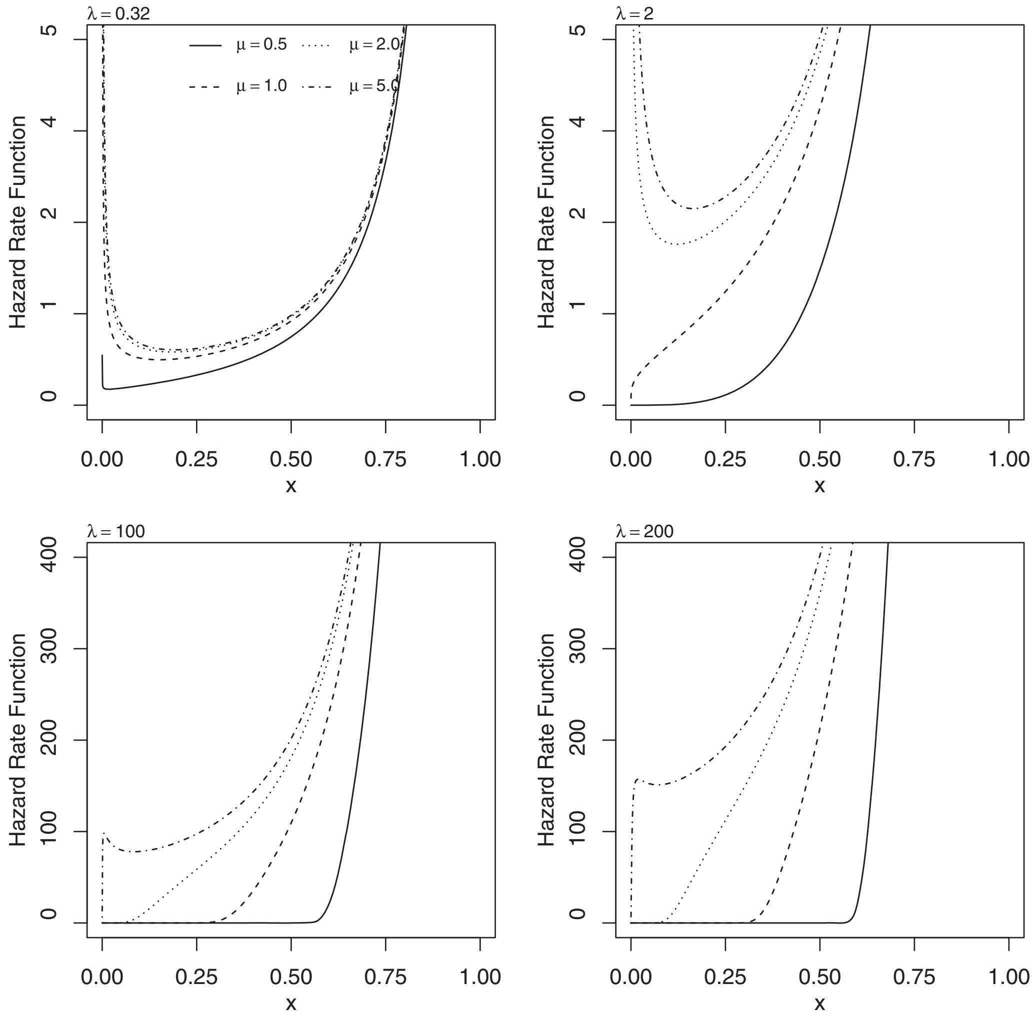

Figure 2. Hazard rate function of UIG distribution for selected values of μ and λ.

In particular, the mean and variance of the UIG distributions, respectively, are given by

$$\mu_1' = \exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1+\frac{2\mu^2}{\lambda}}\right)\right]$$
$$\sigma^2 = \exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1+\frac{4\mu^2}{\lambda}}\right)\right] - \exp\left[\frac{2\lambda}{\mu}\left(1-\sqrt{1+\frac{2\mu^2}{\lambda}}\right)\right]$$

The skewness and kurtosis measures can be obtained from the expressions

$$\text{Skewness} = \frac{\mu_3' - 3\mu_2'\mu_1' + \mu_1'^3}{\sigma^3}$$
$$\text{Kurtosis} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{\sigma^4}$$

upon substituting for the first four raw moments μ_r', $r = 1, 2, 3, 4$.

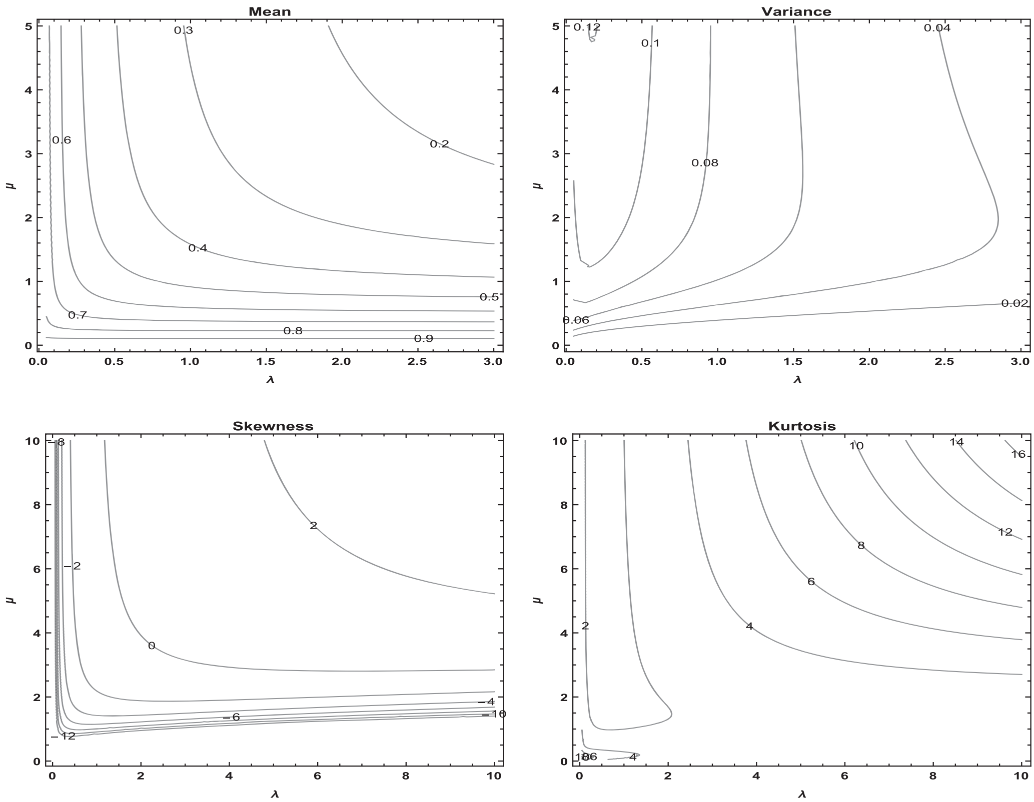

Figure 3. Contour plots of the mean, variance, skewness and kurtosis of UIG distribution.

Figure 3 shows the contours of the mean, variance, skewness and kurtosis of the UIG distribution. This figure shows that the skewness can be negative which is useful in modeling left skewed data.

The first and second raw moments can be used to find the method of moments estimators $\tilde{\mu}$ and $\tilde{\lambda}$ of the parameters μ and λ by solving the equations:

$$\exp\left[\frac{\widetilde{\lambda}}{\widetilde{\mu}}\left(1-\sqrt{1+\frac{2j\widetilde{\mu}^2}{\widetilde{\lambda}}}\right)\right]=m_j, \qquad j=1,2$$

where $m_j = \frac{1}{n}\sum_{i=1}^{n} x_i^j$, $j = 1,2$, are the first two sample moments. It follows that $\widetilde{\lambda} = \frac{l_1 l_2(l_1-l_2)}{2(l_2-2l_1)}$ and $\widetilde{\mu} = \frac{l_1 l_2(l_1-l_2)}{l_2^2-2l_1^2}$ where $l_j = \log(m_j), j = 1,2$. Note that $l_2 < l_1 < 0$, $l_1 - l_2 > 0$ and $l_2-2l_1 = \log(\frac{m_2}{m_1^2}) > 0$. Therefore, $\widetilde{\lambda} > 0$. However, $l_2^2-2l_1^2 = (l_2-\sqrt{2}l_1)(l_2+\sqrt{2}l_1) > 0$ if and only if $l_2-\sqrt{2}l_1 = \log\left(\frac{m_2}{m_1^{\sqrt{2}}}\right) < 0$, i.e. $m_2 < m_1^{\sqrt{2}}$. Therefore, $\widetilde{\mu} > 0$ only if $m_2 < m_1^{\sqrt{2}}$.

In the next section, simple closed form estimates of the parameters μ and λ are obtained by the transformation $Y = -\log X$ and fitting the IG distribution.

5. Estimation using inverse Gaussian distribution

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from UIG distribution with p.d.f. (4). Let $y_i = -\log x_i$, $i = 1, 2, \ldots, n$. Then, $y_1, y_2, \ldots, y_n$ is a random sample from IG distribution with p.d.f. (1).

(a) *Maximum likelihood estimation*

From Seshadri (1999), the distributions of the maximum likelihood estimators (MLEs) of μ and λ, respectively, are given by

$$\widehat{\mu}_{MLE} = \overline{Y}$$
$$\widehat{\lambda}_{MLE} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{Y_i} - \frac{1}{\overline{Y}}\right)}$$

where $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is the sample mean. Note that the MLEs $\widehat{\mu}$ and $\widehat{\lambda}$ are simple and do not require any iterative methods for their calculations.

Also, $\widehat{\mu}$ and $\widehat{\lambda}$ are independent and their respective distributions are given by

$$\widehat{\mu}_{MLE} \sim IG(\mu, n\lambda)$$
$$\widehat{\lambda}_{MLE} \sim \frac{n\lambda}{\chi^2_{(n-1)}}, \qquad n > 1$$

Moreover,

$$E(\widehat{\mu}_{MLE}) = \mu$$
$$E(\widehat{\lambda}_{MLE}) = \frac{n\lambda}{n-3}, \qquad n > 3$$

That is, $\widehat{\mu}_{MLE}$ ($\widehat{\lambda}_{MLE}$) is an unbiased (biased) estimator of μ (λ). An unbiased estimator $\widehat{\widehat{\lambda}}_{MLE}$ of λ is given by

$$\widehat{\widehat{\lambda}}_{MLE} = \frac{n-3}{n}\widehat{\lambda}_{MLE}, \qquad n > 3.$$

The variances of the above MLEs are given by

$$Var(\widehat{\mu}_{MLE}) = \frac{\mu^3}{n\lambda}$$
$$Var(\widehat{\lambda}_{MLE}) = \frac{2\, n^2\lambda^2}{(n-3)^2(n-5)}, \qquad n > 5$$
$$Var(\widehat{\widehat{\lambda}}_{MLE}) = \frac{2\,\lambda^2}{n-5}, \qquad n > 5$$

(b) *Method of moments estimation*

The method of moments estimators (MMEs) of μ and λ are given by

$$\widehat{\mu}_{MME} = \overline{Y}$$
$$\widehat{\lambda}_{MME} = \frac{\overline{Y}^3}{S_y^2}$$

where $S_y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$ is the sample variance.

6. Data analysis

Here, we provide an application of real data set to demonstrate the flexibility and applicability of the proposed distribution over many well known distributions on the unit interval. The data used in this section correspond to the Municipal Human Development Index (MHDI) of the 1188 cities in 2010 located in the the South region of Brazil. They were extracted from the Atlas of Brazil Human Development database available at http://atlasbrasil.org.br/2013/en/. Brazil was one of the first countries to adopt and calculate the HDI for all the municipalities of the country, thus creating the sub-national index - Municipal Human Development Index (MHDI) in 1998. The MHDI adjusts the HDI to the municipal reality and reflects specific and regional challenges in Brazilian human development. To measure the level of human development of municipalities, the MHDI assesses the same dimensions as the global HDI – health, education and income. HDI data from the Brazil were also analyzed by da Paz, Bazán, and Balakrishnan (2016), da Paz, Bazán, and Milan (2017) and da Paz (2017) considering alternatives to Beta distribution.

For this data set, we fit the following two-parameter distributions defined on the unit interval:

(1) Beta distribution:

$$f_1(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \qquad \alpha,\beta>0$$

where $B(\alpha,\beta)$ is the beta function.

(2) Johnson S_B distribution (Johnson 1949):

$$f_2(x) = \frac{\beta}{\sqrt{2\pi}}\,\frac{1}{x\,(1-x)}\,\exp\left\{-\frac{1}{2}\left[\alpha+\beta\log\left(\frac{x}{1-x}\right)\right]^2\right\}, \qquad \alpha\in\mathbb{R},\beta>0$$

(3) Unit-Gamma distribution (Grassia 1977):

$$f_3(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\beta-1}\,(-\log x)^{\alpha-1}, \qquad \alpha,\beta>0$$

where $\Gamma(\alpha)$ is the gamma function.

(4) Kumaraswamy distribution (Kumaraswamy 1980):

$$f_4(x) = \alpha\,\beta\,x^{\alpha-1}\,(1-x^\alpha)^{\beta-1}, \qquad \alpha,\beta>0$$

(5) Unit-Logistic distribution (Tadikamalla, and Johnson 1982):

$$f_5(x) = \frac{\beta\,e^\alpha\,x^{\beta-1}\,(1-x)^{\beta-1}}{\left[x^\beta\,e^\alpha+(1-x)^\beta\right]^2}, \qquad \alpha\in\mathbb{R},\beta>0$$

(6) Complementary Beta distribution (Jones 2002):

$$f_6(x) = B(\alpha,\beta)\,[I_x^{-1}(\alpha,\beta)]^{1-\alpha}\,[1-I_x^{-1}(\alpha,\beta)]^{1-\beta}, \qquad \alpha,\beta>0$$

where $I_x^{-1}(\alpha,\beta)$ is the inverse of the regularized incomplete beta function $I_x(\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)}$, $0<x<1$, in which $B_x(\alpha,\beta)$ is the lower incomplete beta function.

(7) Exponentiated Topp-Leone distribution (Pourdarvish, Mirmostafaee, and Naderi 2015):

$$f_7(x) = 2\,\alpha\,\beta\,(1-x)\,[x\,(2-x)]^{\alpha-1}\left[1-x^\alpha\,(2-x)^\alpha\right]^{\beta-1}, \qquad \alpha,\beta>0$$

(8) Unit-Inverse-Gaussian distribution:

$$f_8(x) = \sqrt{\frac{\beta}{2\pi}} \frac{1}{x(-\log x)^{3/2}} \exp\left[\frac{\beta}{2\alpha^2 \log x}(\log x + \alpha)^2\right], \qquad \alpha, \beta > 0$$

Table 1 shows the maximum likelihood estimates (standard errors) of α and β for all models, where we also have the value of the objective function evaluated at the estimates. From these results we can see that the unit-IG and Johnson S_B have the largest value of the log-likelihood, implying better fit.

In Table 2 we report the test statistics and p-values of three goodness-of-fit tests of the competing distributions.

Among the considered two-parameter distributions on the unit interval, Table 2 shows that the UIG distribution has the smallest test statistic (largest p-value) of the Kolmogorov-Smirnov, Cramér-von Misses and Anderson-Darling goodness-of-fit tests. Thus, we can conclude that the UIG distribution provides the best fit among the distributions considered here. This conclusion is also supported by the histogram-density plots in Figure 4.

The asymptotic distribution of the MLE $\widehat{\boldsymbol{\theta}} = (\widehat{\mu}, \widehat{\lambda})$ of $\boldsymbol{\theta} = (\mu, \lambda)$ is given by

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\text{D}} N_2(\mathbf{0}, \mathbf{I}_1^{-1}(\boldsymbol{\theta}))$$

where $\xrightarrow{\text{D}}$ denotes convergence in distribution, $N_2(\cdot,\cdot)$ denotes the bivariate normal distribution, and

$$\mathbf{I}_1(\boldsymbol{\theta}) = \text{diag}\left(\frac{\lambda}{\mu^3}, \frac{1}{2\lambda^2}\right)$$

is the expected Fisher information matrix about $\boldsymbol{\theta}$ based on a single observation.

For a differentiable function $g(\cdot)$, using the Δ-method, we have

$$\sqrt{n}\left(g(\widehat{\boldsymbol{\theta}}) - g(\boldsymbol{\theta})\right) \xrightarrow{\text{D}} N\left(0, \mathbf{d}^\top(\boldsymbol{\theta})\, \mathbf{I}_1^{-1}(\boldsymbol{\theta})\, \mathbf{d}(\boldsymbol{\theta})\right)$$

Table 1. Parameter estimates (standard errors) of the competing distributions.

Distribution	$\widehat{\alpha}$ (S.E.)	$\widehat{\beta}$ (S.E.)	log-likelihood
Beta	84.1881(3.4536)	33.7066(1.3766)	2099.4284
Johnson S_B	−4.5143(0.0970)	4.8839(0.1002)	2100.5994
Unit-Gamma	33.3926(1.3633)	98.6669(4.0587)	2099.3503
Kumaraswamy	18.8766(0.4075)	348.3218(41.8774)	2071.5714
Unit-Logistic	−7.9124(0.1970)	8.5468(0.2052)	2090.5050
Complementary Beta	0.0371(0.0009)	0.0929(0.0023)	2093.4244
Exponentiated Topp-Leone	44.1611(1.0460)	26.7063(2.0359)	2095.0692
Unit-IG	0.3384(0.0017)	11.0346(0.4528)	2100.1204

Table 2. Test statistics (p-values) of three goodness-of-fit tests of the competing distributions.

Distribution	Goodness-of-fit test		
	Kolmogorov–Smirnov	Cramér-von Misses	Anderson–Darling
Beta	0.0285 (0.2902)	0.1144 (0.5187)	0.7676 (0.5050)
Johnson S_B	0.0248 (0.4598)	0.0651 (0.7818)	0.4466 (0.8016)
Unit-Gamma	0.0285 (0.2876)	0.1164 (0.5104)	0.7814 (0.4946)
Kumaraswamy	0.0428 (0.0259)	0.4773 (0.0455)	3.4865 (0.0156)
Unit-Logistic	0.0370 (0.0772)	0.1837 (0.3015)	1.3954 (0.2036)
Complementary Beta	0.0361 (0.0898)	0.1647 (0.3478)	1.3281 (0.2234)
Exponentiated Topp-Leone	0.0269 (0.3585)	0.1190 (0.4997)	0.8051 (0.4773)
Unit-IG	0.0208 (0.6817)	0.0475 (0.8915)	0.3219 (0.9208)

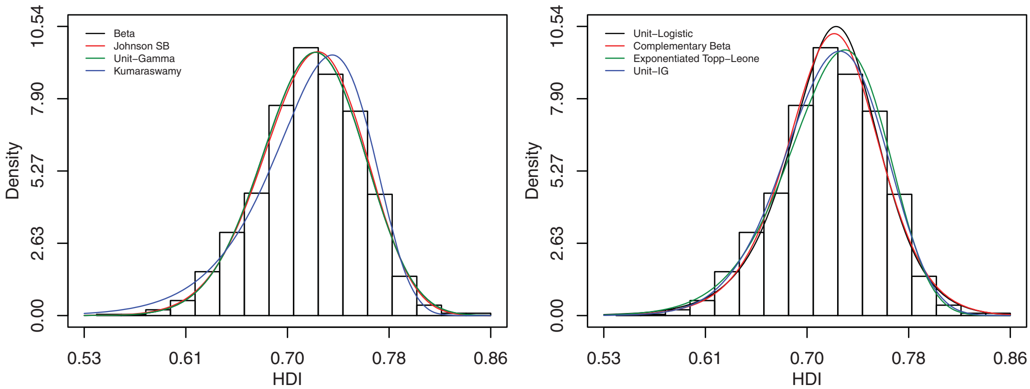

Figure 4. Histogram and density curves of the competing distributions.

where $\top$ denotes the transpose of a vector and

$$\mathbf{d}^\top(\boldsymbol{\theta}) = (d_1(\boldsymbol{\theta}), d_2(\boldsymbol{\theta})) = \left(\frac{\partial g(\boldsymbol{\theta})}{\partial \mu}, \frac{\partial g(\boldsymbol{\theta})}{\partial \lambda}\right)$$

This asymptotic result can be used to make statistical inference about functions of the population parameters. For example, let

$$\mu_{MHDI} = g(\boldsymbol{\theta}) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2}{\lambda}}\right)\right]$$

be the average MHDI index of the population of all cities in South Brazil. For our data, the MLE of μ_{MHDI} is $\widehat{\mu}_{MHDI} = g(\widehat{\boldsymbol{\theta}}) = 0.7141$ with estimated standard error (S.E.)

$$\widehat{S.E.}(\widehat{\mu}_{MHDI}) = \sqrt{\frac{1}{n}\left[d_1^2(\widehat{\boldsymbol{\theta}})\ \frac{\widehat{\mu}^3}{\widehat{\lambda}} + d_2^2(\widehat{\boldsymbol{\theta}})\ 2\widehat{\lambda}^2\right]} = 0.0012$$

Now a 95% confidence interval of μ_{MHDI} is

$$\widehat{\mu}_{MHDI} \pm 1.96\ \widehat{S.E.}(\widehat{\mu}_{MHDI}) \equiv (0.7117, 0.7165)$$

Hypothesis testing about μ_{MHDI} can also be performed. For example, for testing

$$H_0 : \mu_{MHDI} \leq 0.71 \qquad \text{versus} \qquad H_1 : \mu_{MHDI} > 0.71$$

the test statistic is

$$z = \frac{\widehat{\mu}_{MHDI} - 0.71}{\widehat{S.E.}(\widehat{\mu}_{MHDI})} = 3.4167$$

with p-value of the test $P(Z > 3.4167) = 0.0003$, rejecting H_0 at significance level $\alpha = 0.05$.

Finally, we compare the average MHDI of South region to other regions (North, Northeast, Central-West, Southeast) of Brazil as well as to the grand average of Brazil. Table 3 shows the point estimates and confidence intervals of the average MHDI of all regions of Brazil, using the UIG distribution. This table also show that South region has the best average MHDI among all regions of Brazil.

Table 3. Point estimate (S.E.) and confidence interval of μ_{MHDI} by region.

Region	$\widehat{\mu}_{MHDI}$ (S.E.)	95% C.I. of μ_{MHDI}
South	0.7141 (0.0012)	(0.7117, 0.7165)
Southeast	0.6990 (0.0013)	(0.6965, 0.7015)
North	0.6081 (0.0029)	(0.6024, 0.6138)
Northeast	0.5909 (0.0011)	(0.5887, 0.5931)
Central-West	0.6895 (0.0017)	(0.6862, 0.6928)
Brazil	0.6594 (0.0010)	(0.6574, 0.6614)

7. Concluding remarks

A new two-parameter distribution over the unit interval, called the Unit-Inverse Gaussian distribution, is introduced and studied in detail. Unlike other distributions on the unit interval, the maximum likelihood estimates of the parameters are expressed in simple closed forms which do not need iterative methods to compute. Application of the proposed distribution to a real data set shows better fit than many known two-parameter distributions on the unit interval, such as Beta, Johnson S_B, Unit-Gamma, Unit-Logistic and Kumaraswamy distributions. We hope that our approach will be found useful to the data analysts seeking more appropriate models on the unit interval.

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