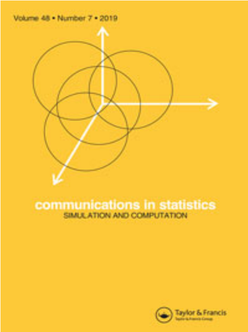

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Bias-corrected maximum likelihood estimators of the parameters of the inverse Weibull distribution

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ABSTRACT
Maximum likelihood estimators usually have biases of the order $\mathcal{O}(n^{-1})$ for large sample size n which are very often ignored because of the fact that they are small when compared to the standard errors of the parameter estimators that are of order $\mathcal{O}(n^{-1/2})$. The accuracy of the estimates may be affected by such bias. To reduce such bias of the MLEs from order $\mathcal{O}(n^{-1})$ to order $\mathcal{O}(n^{-2})$, we adopt some bias-corrected techniques. In this paper, we adopt two approaches to derive first-order bias corrections for the the maximum likelihood estimators of the parameters of the Inverse Weibull distribution. The first one is the analytical methodology suggested by Cox and Snell (1968) and the second is based on the parametric Bootstrap resampling method. Monte Carlo simulations are conducted to investigate the performance of these methodologies. Our results reveal that the bias corrections improve the accuracy as well as the consistency of the estimators. Finally, an example with a real data set is presented.

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1. Introduction

The Weibull distribution has been adopted as a successful model for many product failure mechanisms because of its flexibility and wide range of applicability. Extensive work has been done on this distribution, both from the frequentist and Bayesian points of view, see for example the excellent review by Johnson, Kotz, and Balakrishnan (1995) and Kundu (2008), for some references. The limitation of this model is its incapability to accommodate non-monotone hazard rates such as bathtub shape, the unimodal (upside-down bathtub) or modified unimodal shape which are common in human mortality, machine life cycles, biological and medical studies. Therefore, if the empirical study indicates that the hazard function of the underlying distribution is not monotone, and it is unimodal, then inverse Weibull (IW) distribution may be used to analyze such data. Extensive work has been done on the IW distribution, see for example Keller and Kanath (1982), Calabria and Pulcini (1989), Erto (1989), Calabria and Pulcini (1990), Calabria and Pulcini (1992), Jiang, Ji, and Xiao (2003), Mahmoud, Sultan, and Amer (2003), Maswadah (2003), Kundu and Howlader (2010), Nassar and Abo-Kasem (2017) and the references cited therein. The Inverse Weibull also known as the Fréchet distribution is named after French mathematician Maurice René Fréchet, who developed it in the 1920s as a maximum value distribution, which is also known as the extreme

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value distribution of type II (Fréchet 1928). The IW distribution has the ability to model failure rates which are quite common in reliability and biological studies. Kotz and Nadarajah (2000) in their book, described this distribution and discussed its wide applicability in different spheres such as accelerated life testing, natural calamities, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, track race records and so on. The IW distribution is also a limiting distribution of the largest order statistics.

The random variable X has Inverse Weibull distribution if its cumulative distribution function (c.d.f.) is defined as:

$$F(x \mid \mu, \beta) = \exp\left[-\left(\frac{\mu}{x}\right)^{\beta}\right], \quad x > 0 \tag{1}$$

where $\mu > 0$ and $\beta > 0$ are scale and shape parameters, respectively. The corresponding probability density function (p.d.f.) is given by:

$$f(x \mid \mu, \beta) = \beta\, \mu^{\beta}\, x^{-(\beta+1)} \exp\left[-\left(\frac{\mu}{x}\right)^{\beta}\right]. \tag{2}$$

If $\beta = 1$, the p.d.f. of IW becomes the inverse exponential distribution, and when $\beta = 2$, the IW p.d.f. is referred to as the inverse Rayleigh distribution.

Many authors have discussed the situations where the data shows the upside-down bathtub shape hazard rates. For example: Efron (1988) analyzed the data set in the context of head and neck cancer, in which the hazard rate initially increased, attained a maximum and then decreased before it stabilized owing to a therapy. Bennett (1983) analyzed lung cancer trial data which showed that failure rates were unimodal in nature. Langlands et al. (1979) have studied the breast carcinoma data and found that the mortality reached a peak after some finite period, and then declined gradually.

Choice of estimation methodology is important when estimating parameters from any probability distribution. Among all the classical estimation methods, the most frequently used method is the maximum likelihood estimation method (Pawitan 2001). Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. However, it is well known that in finite samples the maximum likelihood estimator (MLE) not possess any desirable sampling properties. In particular, the MLE is often biased for small sample sizes. The determination of such bias can be complicated, as the likelihood equations (first-order conditions) that determine the maximum of the likelihood function are often highly non-linear, and do not possess a closed-form solution. Therefore, it is important to derive closed-form expressions for the first-order biases of estimators in some classes of models which can be used in practical applications in order to evaluate the accuracy of these estimators and also to define estimators with smaller biases. Several researchers strive to develop nearly unbiased estimators for the parameters of several distributions. Readers may refer to Cribari-Neto and Vasconcellos (2002), Saha and Paul (2005), Lemonte, Cribari-Neto, and Vasconcellos (2007), Giles (2012), Schwartz, Godwin, and Giles (2013), Giles, Feng, and Godwin (2013), Ling and Giles (2014), Schwartz and Giles (2016), Wang and Wang (2017), Mazucheli and Dey (2018), Mazucheli, Menezes, and Dey (2017a), Mazucheli, Menezes, and Nadarajah (2017b), Reath, Dong, and Wang (2018) and references cited therein.

In this paper, we adopt two bias corrected techniques to reduce the bias of the MLE from order $\mathcal{O}(n^{-1})$ to order $\mathcal{O}(n^{-2})$ for the two parameter IW distribution and illustrate their performance. First, we discuss the analytical methodology suggested by Cox and Snell (1968), which is called "corrective" approach and derive "bias adjusted" MLEs of second order where the bias-correction is done by subtracting the bias (estimated at the MLEs of the parameters) from the true value of the MLEs. Secondly, as an alternative to the analytically bias-corrected MLEs, we consider the bias-corrected MLEs through Efron (1982) parametric Bootstrap resampling method which is also second-order bias correction. In this method bias correction is performed numerically without deriving analytical expression for the bias function. In this paper, we are focusing on corrective approach for bias correction and thus we have not attempted the preventive approach suggested by Firth (1993). The effectiveness of the suggested two bias correction techniques, in terms of both bias reduction and its impact on root mean squared error, is compared with classical MLEs. The simulation study of the proposed bias corrected estimators reveal that these estimators are quite accurate even for small sample sizes and are superior to classical MLEs in terms of their bias and root mean squared errors.

The remainder of this paper is organized as follows. In Sections 2 and 3, we summarize the maximum likelihood estimators and its bias-corrected version. A Monte Carlo simulation experiment are conducted in Section 4 to compare the performance among the Cox-Snell bias adjusted estimators, bootstrap-based bias-adjusted estimators and the uncorrected MLEs. In Section 5, an application to real data set is presented for illustrative purposes. Finally, Section 6 concludes the paper.

2. Maximum likelihood estimation

In this section, we discuss the maximum likelihood estimators (MLEs) of the inverse Weibull distribution and their asymptotic properties.

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a random sample from the Inverse Weibull distribution with p.d.f. (1). Then, the log- likelihood function for $\boldsymbol{\Theta} = (\mu, \beta)$ can be written as:

$$\ell(\boldsymbol{\Theta} \mid \boldsymbol{x}) = n \log \beta + n \beta \log \mu - (\beta + 1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\mu}{x_i}\right)^{\beta}. \tag{3}$$

The MLE of μ and β may be found by solving the following non-linear equations:

$$\frac{\partial}{\partial \mu} \ell(\boldsymbol{\Theta} \mid \boldsymbol{x}) = \frac{n \beta}{\mu} - \frac{\beta}{\mu} \sum_{i=1}^{n} \left(\frac{\mu}{x_i}\right)^{\beta} = 0, \tag{4}$$

$$\frac{\partial}{\partial \beta} \ell(\boldsymbol{\Theta} \mid \boldsymbol{x}) = \frac{n}{\beta} + n \log \mu - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\mu}{x_i}\right)^{\beta} \log\left(\frac{\mu}{x_i}\right) = 0. \tag{5}$$

The expected Fisher information matrix is given by:

$$\boldsymbol{I}(\boldsymbol{\Theta} \mid \boldsymbol{x}) = n \begin{bmatrix} \dfrac{\pi^2 + 6\gamma^2 - 12\gamma + 6}{6 \beta^2} & \dfrac{1-\gamma}{\mu} \\ \dfrac{1-\gamma}{\mu} & \dfrac{\beta^2}{\mu^2} \end{bmatrix}. \tag{6}$$

Then, its inverse is defined as:

$$\boldsymbol{I}^{-1}(\boldsymbol{\Theta} \mid \boldsymbol{x}) = \frac{1}{n} \begin{bmatrix} \dfrac{6\beta^2}{\pi^2} & \dfrac{6 \mu (\gamma - 1)}{\pi^2} \\ \dfrac{6 \mu (\gamma - 1)}{\pi^2} & \dfrac{\mu^2(\pi^2 + 6\gamma^2 - 12\gamma + 6)}{\beta^2 \pi^2} \end{bmatrix} \tag{7}$$

where $\pi \simeq 3.141593$ and $\gamma \simeq 0.577216$ is the Euler's constant.

3. Bias-corrected MLEs

In this section, we describe the two "corrective" approach to bias-correction, both of which reduces the biases of the MLEs of inverse Weibull distribution to the second order magnitude. We consider the methodology introduced by Cox and Snell (1968) and the parametric Bootstrap resampling method (Efron 1982).

3.1. Cox-snell methodology

Let $\ell(\boldsymbol{\Theta} \mid \boldsymbol{x})$ denote the log-likelihood function of a p-dimensional parameter vector $\boldsymbol{\Theta}$ based on a sample of observations $\boldsymbol{x}$. We shall assume some regularity conditions on the behavior of the log-likelihood function (see Cox and Hinkley, 1974).

The joint cumulants of the derivatives of ℓ are given by:

$$\boldsymbol{I}_{ij} = \mathbb{E}\left[\frac{\partial^2 \ell}{\partial \boldsymbol{\Theta}_i \, \partial \boldsymbol{\Theta}_j}\right], \quad \boldsymbol{I}_{ijl} = \mathbb{E}\left[\frac{\partial^3 \ell}{\partial \boldsymbol{\Theta}_i \, \partial \boldsymbol{\Theta}_j \, \partial \boldsymbol{\Theta}_l}\right] \quad \text{and} \quad \boldsymbol{I}_{ij,l} = \mathbb{E}\left[\left(\frac{\partial^2 \ell}{\partial \boldsymbol{\Theta}_i \, \partial \boldsymbol{\Theta}_j}\right)\left(\frac{\partial \ell}{\partial \boldsymbol{\Theta}_l}\right)\right].$$

for $i, j, l = 1, \ldots, p$. All expression above are assumed to be to the order $\mathcal{O}(n)$.

Cox and Snell (1968) showed that when the sample data are independent, but not necessarily identically distributed, the bias of the $r\text{–}th$ element of the MLE of $\boldsymbol{\Theta}$, $\widehat{\boldsymbol{\Theta}}$, can be expressed as:

$$\mathcal{B}(\widehat{\boldsymbol{\Theta}}_r) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \boldsymbol{I}^{si}\boldsymbol{I}^{jl}\left[0.5\boldsymbol{I}_{ijl} + \boldsymbol{I}_{ij,l}\right] + \mathcal{O}(n^{-2}), \tag{8}$$

where $r = 1, \ldots, p$ and $\boldsymbol{I}^{ij}$ denotes the the $(i, j)\text{–}th$ element of the inverse of the expected Fisher information matrix.

After extensive algebraic manipulation, we are able to obtain the following expressions for the Inverse Weibull distribution:

$$\begin{aligned}
\boldsymbol{I}_{111} &= \frac{n}{2\,\beta^3}\left[4\,\xi(3) + (\gamma-1)\,(\pi^2 + 2\,\gamma^2 - 4\,\gamma - 4)\right],\\
\boldsymbol{I}_{112} = \boldsymbol{I}_{121} = \boldsymbol{I}_{211} &= -\frac{n}{6\,\mu\,\beta}\,[\pi^2 + 6\,\gamma^2 - 24\,\gamma + 12],\\
\boldsymbol{I}_{122} = \boldsymbol{I}_{122} = \boldsymbol{I}_{221} &= \frac{n}{\mu^2}\,[\beta\,(\gamma-3) - \gamma + 1],\\
\boldsymbol{I}_{222} &= -\frac{n\,\beta^2\,(\beta-3)}{\mu^3}.
\end{aligned}$$

We also have:

$$\begin{aligned}
\boldsymbol{I}_{11,1} &= -\frac{n}{2\,\beta^3}\left[\pi^2\,\gamma + 2\,\gamma^3 - \frac{5}{3}\,\pi^2 - 10\,\gamma^2 + 8\,\gamma + 4\,\xi(3)\right],\\
\boldsymbol{I}_{11,2} = \boldsymbol{I}_{12,1} = \boldsymbol{I}_{21,1} &= \frac{n}{6\,\mu\,\beta}\left[\pi^2 + 6\,\gamma^2 - 24\,\gamma + 12\right],\\
\boldsymbol{I}_{12,2} = \boldsymbol{I}_{21,2} &= \frac{n\,\beta\,(3-\gamma)}{\mu^2},\\
\boldsymbol{I}_{22,1} &= -\frac{n\,(\beta-1)\,(\gamma-1)}{\mu^2},\\
\boldsymbol{I}_{22,2} &= \frac{n\,(\beta-1)\,\beta^2}{\mu^3},
\end{aligned}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann's Zeta function.

Thus, the second-order bias of the maximum likelihood estimator of μ and β are defined, respectively, as:

$$\mathcal{B}(\widehat{\mu}) = \frac{1.379530691\,\beta}{n} \tag{9}$$

and:

$$\mathcal{B}(\widehat{\beta}) = \frac{\mu\ (0.3698145391\,\beta + 0.5543324494)}{n\beta^2}. \tag{10}$$

Using (9) and (10) we define the bias-corrected estimators (BCE) of $\widehat{\mu}$ and $\widehat{\beta}$ to be, respectively $\widehat{\mu}_{BCE} = \widehat{\mu} - \widehat{\mathcal{B}}(\widehat{\mu})$ and $\widehat{\beta}_{BCE} = \widehat{\beta} - \widehat{\mathcal{B}}(\widehat{\beta})$. It is to be noted that $\widehat{\mu}_{BCE}$ and $\widehat{\beta}_{BCE}$ have bias of order $\mathcal{O}(n^{-2})$ and is expected that they have superior sampling properties relative to $\widehat{\mu}$ and $\widehat{\beta}$.

3.2. Parametric bootstrap

An alternative approach to obtain bias-corrected estimators is the parametric Bootstrap resampling method (PBE), pioneered by Efron (1982). In particular, this method uses the MLEs of the data to generate pseudo-random samples from the distribution to estimate the bias and then subtract the bias from the MLE.

Table 1. Estimated bias (root mean-squared error) for μ and β, ($\mu = 0.5$).

		Estimator of μ			Estimator of β		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.1785 (0.6171)	−0.0243 (0.3870)	−0.0308 (0.5086)	0.0866 (0.1984)	0.0057 (0.1539)	−0.0131 (0.1492)
	20	0.0777 (0.3182)	−0.0083 (0.2519)	−0.0191 (0.2537)	0.0380 (0.1101)	0.0009 (0.0962)	−0.0029 (0.0956)
	30	0.0499 (0.2339)	−0.0044 (0.2011)	−0.0092 (0.1989)	0.0246 (0.0826)	0.0005 (0.0752)	−0.0011 (0.0750)
	40	0.0368 (0.1919)	−0.0029 (0.1717)	−0.0055 (0.1709)	0.0181 (0.0694)	0.0002 (0.0647)	−0.0007 (0.0646)
	50	0.0298 (0.1676)	−0.0016 (0.1532)	−0.0032 (0.1526)	0.0144 (0.0606)	0.0002 (0.0573)	−0.0004 (0.0572)
1.0	10	0.0539 (0.2122)	0.0047 (0.1803)	−0.0018 (0.1779)	0.1651 (0.3798)	0.0044 (0.2949)	−0.0322 (0.2866)
	20	0.0261 (0.1338)	0.0022 (0.1233)	0.0008 (0.1228)	0.0727 (0.2170)	−0.0013 (0.1904)	−0.0089 (0.1892)
	30	0.0171 (0.1060)	0.0014 (0.1003)	0.0007 (0.1002)	0.0484 (0.1660)	0.0002 (0.1515)	−0.0030 (0.1510)
	40	0.0125 (0.0891)	0.0008 (0.0855)	0.0005 (0.0855)	0.0364 (0.1400)	0.0006 (0.1305)	−0.0011 (0.1304)
	50	0.0096 (0.0781)	0.0003 (0.0756)	0.0001 (0.0756)	0.0294 (0.1217)	0.0010 (0.1149)	−0.0001 (0.1149)
1.5	10	0.0267 (0.1270)	0.0022 (0.1157)	−0.0001 (0.1149)	0.2551 (0.5962)	0.0130 (0.4647)	−0.0414 (0.4551)
	20	0.0128 (0.0836)	0.0006 (0.0797)	0.0000 (0.0797)	0.1186 (0.3326)	0.0069 (0.2894)	−0.0044 (0.2874)
	30	0.0079 (0.0659)	−0.0003 (0.0639)	−0.0005 (0.0639)	0.0741 (0.2475)	0.0017 (0.2253)	−0.0030 (0.2249)
	40	0.0056 (0.0566)	−0.0005 (0.0553)	−0.0007 (0.0553)	0.0574 (0.2061)	0.0037 (0.1911)	0.0011 (0.1909)
	50	0.0045 (0.0503)	−0.0004 (0.0494)	−0.0004 (0.0494)	0.0438 (0.1801)	0.0013 (0.1699)	−0.0004 (0.1698)
2.0	10	0.0165 (0.0929)	0.0009 (0.0872)	−0.0003 (0.0869)	0.3341 (0.7746)	0.0121 (0.6025)	−0.0558 (0.6028)
	20	0.0075 (0.0618)	−0.0004 (0.0598)	−0.0007 (0.0597)	0.1478 (0.4323)	−0.0003 (0.3783)	−0.0153 (0.3762)
	30	0.0051 (0.0493)	−0.0003 (0.0482)	−0.0004 (0.0482)	0.0952 (0.3310)	−0.0011 (0.3024)	−0.0075 (0.3016)
	40	0.0038 (0.0425)	−0.0002 (0.0418)	−0.0003 (0.0418)	0.0710 (0.2738)	−0.0005 (0.2553)	−0.0039 (0.2551)
	50	0.0031 (0.0379)	−0.0002 (0.0374)	−0.0002 (0.0374)	0.0561 (0.2398)	−0.0006 (0.2267)	−0.0027 (0.2267)
3.0	10	0.0108 (0.0603)	0.0020 (0.0575)	0.0014 (0.0574)	0.5055 (1.1442)	0.0219 (0.8851)	−0.0335 (0.9758)
	20	0.0052 (0.0404)	0.0007 (0.0394)	0.0006 (0.0394)	0.2267 (0.6551)	0.0041 (0.5722)	−0.0180 (0.5703)
	30	0.0032 (0.0324)	0.0002 (0.0319)	0.0001 (0.0319)	0.1447 (0.4968)	0.0001 (0.4534)	−0.0096 (0.4523)
	40	0.0021 (0.0282)	−0.0002 (0.0279)	−0.0002 (0.0279)	0.1115 (0.4121)	0.0042 (0.3830)	−0.0010 (0.3824)
	50	0.0017 (0.0252)	−0.0001 (0.0249)	−0.0002 (0.0250)	0.0890 (0.3605)	0.0038 (0.3397)	0.0005 (0.3394)
5.0	10	0.0051 (0.0348)	0.0005 (0.0337)	−0.0003 (0.0338)	0.6721 (1.5873)	−0.1104 (1.2445)	0.0851 (1.5757)
	20	0.0025 (0.0243)	0.0002 (0.0239)	0.0000 (0.0239)	0.3627 (1.0698)	−0.0072 (0.9370)	0.0013 (1.0129)
	30	0.0015 (0.0197)	−0.0001 (0.0195)	−0.0001 (0.0195)	0.2383 (0.8317)	−0.0026 (0.7602)	−0.0104 (0.7788)
	40	0.0011 (0.0168)	−0.0001 (0.0167)	−0.0001 (0.0167)	0.1765 (0.6905)	−0.0020 (0.6445)	−0.0092 (0.6486)
	50	0.0008 (0.0148)	−0.0001 (0.0147)	−0.0001 (0.0148)	0.1384 (0.6015)	−0.0034 (0.5692)	−0.0086 (0.5695)

For a parameter vector $\boldsymbol{\Theta}$ the estimated bias of $\widehat{\boldsymbol{\Theta}}$ is defined as:

$$\widehat{\mathcal{B}}(\widehat{\boldsymbol{\Theta}}) = \frac{1}{B}\sum_{j=1}^{B} \boldsymbol{\Theta}_{(j)} - \widehat{\boldsymbol{\Theta}}, \tag{11}$$

where $\widehat{\boldsymbol{\Theta}}_{(j)}$ is the MLE of $\boldsymbol{\Theta}$ obtained from the j-th Bootstrap sample, generated from (2) and using the maximum likelihood estimate $\widehat{\boldsymbol{\Theta}}$ as the true value. Thus, the Bootstrap bias-corrected estimator is:

$$\widehat{\boldsymbol{\Theta}}_{PBE} = 2\,\widehat{\boldsymbol{\Theta}} - \frac{1}{B}\sum_{j=1}^{B} \widehat{\boldsymbol{\Theta}}_{(j)}. \tag{12}$$

It is noteworthy that the PBE does not involve analytical derivatives. Also, this approach provides an unbiased estimator to second order.

4. Simulation study

In this section, we carry out Monte Carlo simulations in order to compare the finite-sample behavior of the MLEs of μ and β and their bias corrections proposed in Section 3. The comparison is based on the empirical biases and root mean squared error criteria.

We have taken sample sizes $n = \{10, 20, 30, 40, 50\}$ and the parameters are fixed at $\mu = \{0.5, 1.0, 1.5, 2.0\}$ and $\beta = \{0.5, 1.0, 1.5, 2.0, 3.0, 5.0\}$. For each combination of n, μ and

Table 2. Estimated bias (root mean-squared error) for μ and β, ($\mu = 1.0$).

		Estimator of μ			Estimator of β		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.3330 (1.1055)	−0.0617 (0.7143)	0.0259 (1.0328)	0.0869 (0.1985)	0.0060 (0.1540)	−0.0135 (0.1499)
	20	0.1544 (0.6298)	−0.0173 (0.4993)	−0.0259 (0.5373)	0.0380 (0.1101)	0.0009 (0.0962)	−0.0030 (0.0956)
	30	0.0998 (0.4677)	−0.0088 (0.4021)	−0.0166 (0.4049)	0.0246 (0.0826)	0.0005 (0.0752)	−0.0012 (0.0750)
	40	0.0736 (0.3838)	−0.0058 (0.3435)	−0.0110 (0.3414)	0.0181 (0.0694)	0.0002 (0.0647)	−0.0007 (0.0646)
	50	0.0596 (0.3351)	−0.0031 (0.3064)	−0.0063 (0.3058)	0.0144 (0.0606)	0.0002 (0.0573)	−0.0004 (0.0572)
1.0	10	0.0998 (0.4247)	0.0022 (0.3634)	−0.0087 (0.3648)	0.1705 (0.3902)	0.0090 (0.3026)	−0.0278 (0.2935)
	20	0.0463 (0.2616)	−0.0009 (0.2416)	−0.0037 (0.2405)	0.0765 (0.2160)	0.0023 (0.1881)	−0.0054 (0.1868)
	30	0.0309 (0.2055)	−0.0004 (0.1948)	−0.0017 (0.1945)	0.0482 (0.1632)	0.0000 (0.1488)	−0.0032 (0.1484)
	40	0.0218 (0.1744)	−0.0015 (0.1677)	−0.0021 (0.1677)	0.0367 (0.1386)	0.0010 (0.1291)	−0.0008 (0.1288)
	50	0.0167 (0.1535)	−0.0019 (0.1488)	−0.0023 (0.1487)	0.0290 (0.1211)	0.0006 (0.1143)	−0.0005 (0.1143)
1.5	10	0.0545 (0.2574)	0.0052 (0.2348)	0.0008 (0.2332)	0.2420 (0.5699)	0.0017 (0.4448)	−0.0526 (0.4359)
	20	0.0267 (0.1673)	0.0021 (0.1592)	0.0010 (0.1591)	0.1085 (0.3267)	−0.0025 (0.2869)	−0.0138 (0.2853)
	30	0.0184 (0.1343)	0.0019 (0.1298)	0.0014 (0.1298)	0.0675 (0.2453)	−0.0046 (0.2251)	−0.0093 (0.2247)
	40	0.0138 (0.1142)	0.0014 (0.1113)	0.0011 (0.1113)	0.0495 (0.2037)	−0.0040 (0.1909)	−0.0065 (0.1908)
	50	0.0115 (0.1013)	0.0016 (0.0991)	0.0015 (0.0992)	0.0387 (0.1783)	−0.0038 (0.1693)	−0.0054 (0.1693)
2.0	10	0.0328 (0.1856)	0.0015 (0.1737)	−0.0008 (0.1731)	0.3385 (0.7688)	0.0159 (0.5953)	−0.0518 (0.5970)
	20	0.0149 (0.1226)	−0.0009 (0.1187)	−0.0015 (0.1187)	0.1531 (0.4402)	0.0046 (0.3842)	−0.0105 (0.3816)
	30	0.0096 (0.0984)	−0.0010 (0.0963)	−0.0012 (0.0963)	0.0983 (0.3311)	0.0018 (0.3016)	−0.0047 (0.3008)
	40	0.0078 (0.0851)	−0.0002 (0.0836)	−0.0004 (0.0836)	0.0740 (0.2749)	0.0025 (0.2557)	−0.0010 (0.2553)
	50	0.0055 (0.0753)	−0.0009 (0.0744)	−0.0010 (0.0744)	0.0592 (0.2420)	0.0024 (0.2282)	0.0003 (0.2280)
3.0	10	0.0199 (0.1190)	0.0024 (0.1137)	0.0011 (0.1135)	0.5080 (1.1583)	0.0240 (0.8977)	−0.0340 (0.9783)
	20	0.0097 (0.0808)	0.0007 (0.0790)	0.0004 (0.0790)	0.2269 (0.6670)	0.0043 (0.5840)	−0.0178 (0.5823)
	30	0.0064 (0.0650)	0.0003 (0.0640)	0.0002 (0.0640)	0.1438 (0.4943)	−0.0008 (0.4512)	−0.0104 (0.4500)
	40	0.0046 (0.0560)	0.0000 (0.0554)	0.0000 (0.0554)	0.1031 (0.4078)	−0.0039 (0.3810)	−0.0090 (0.3808)
	50	0.0039 (0.0499)	0.0002 (0.0495)	0.0002 (0.0495)	0.0822 (0.3579)	−0.0029 (0.3387)	−0.0062 (0.3385)
5.0	10	0.0103 (0.0702)	0.0011 (0.0680)	−0.0007 (0.0681)	0.6668 (1.5825)	−0.1150 (1.2425)	0.1084 (1.6167)
	20	0.0056 (0.0489)	0.0010 (0.0481)	0.0006 (0.0481)	0.3680 (1.0721)	−0.0023 (0.9375)	0.0134 (1.0272)
	30	0.0036 (0.0391)	0.0005 (0.0386)	0.0004 (0.0386)	0.2355 (0.8243)	−0.0053 (0.7536)	−0.0131 (0.7716)
	40	0.0026 (0.0336)	0.0002 (0.0333)	0.0002 (0.0334)	0.1760 (0.6940)	−0.0025 (0.6482)	−0.0100 (0.6520)
	50	0.0020 (0.0298)	0.0001 (0.0296)	0.0001 (0.0296)	0.1411 (0.6139)	−0.0007 (0.5810)	−0.0057 (0.5820)

β, we simulate pseudo-random samples from inverse Weibull distribution using the inverse transform method, that is $\boldsymbol{x} = (x_1, \ldots, x_n)$ is generated from:

$$x_i = \exp\left[\frac{\beta \log\mu - \log(-\log u_i)}{\beta}\right], \quad i = 1, \ldots, n$$

where u_i are random numbers from a standard uniform distribution.

The number of Monte Carlo replications are fixed at $M = 10,000$ and $B = 1000$ Bootstrap replicates. All simulation studies are carried out in Ox Console (Doornik, 2007) and the MaxBFGS function is used to obtain the maximum likelihood estimates. The results are reported in Tables 1–4.

From Table 1, we observe that the biases of μ_{BCE}, μ_{PBE} and β_{BCE}, β_{PBE} are consistently smaller than the biases of μ_{MLE} and β_{MLE}; the biases of MLEs appear positive for both the parameters; the magnitude of the biases of μ and β appears largest for the unadjusted ML estimates for every n; in most of the cases, the biases of all the estimators of μ and β generally approach to zero as n increases; Similar results are observed in Tables 2–4. The estimators μ_{BCE}, β_{BCE} and μ_{PBE}, β_{PBE} clearly outperform the estimators μ_{MLE} and β_{MLE} as far as bias goes. Thus the estimators based on analytical and parametric Bootstrap resampling methods achieve substantial bias reduction, especially for the small and moderate sample sizes and therefore we consider them as better alternatives of the MLEs for μ and β. We also observe that the bias-corrected estimates are closer to the true parameter values than the unadjusted estimates as sample size increases. Additionally, the root mean squared errors of the corrected

Table 3. Estimated bias (root mean-squared error) for μ and β, ($\mu = 1.5$).

		Estimator of μ			Estimator of β		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.4467 (1.4789)	−0.1248 (0.9730)	0.1028 (1.4776)	0.0875 (0.1988)	0.0064 (0.1540)	−0.0137 (0.1503)
	20	0.2296 (0.9334)	−0.0274 (0.7416)	−0.0071 (0.8697)	0.0381 (0.1101)	0.0010 (0.0962)	−0.0031 (0.0959)
	30	0.1497 (0.7016)	−0.0132 (0.6032)	−0.0176 (0.6313)	0.0246 (0.0826)	0.0005 (0.0752)	−0.0012 (0.0751)
	40	0.1104 (0.5757)	−0.0087 (0.5152)	−0.0149 (0.5176)	0.0181 (0.0694)	0.0002 (0.0647)	−0.0007 (0.0646)
	50	0.0893 (0.5027)	−0.0047 (0.4596)	−0.0093 (0.4591)	0.0144 (0.0606)	0.0002 (0.0573)	−0.0004 (0.0572)
1.0	10	0.1520 (0.6272)	0.0070 (0.5399)	−0.0042 (0.5530)	0.1705 (0.3855)	0.0090 (0.2982)	−0.0281 (0.2887)
	20	0.0737 (0.3934)	0.0030 (0.3631)	−0.0011 (0.3618)	0.0794 (0.2196)	0.0049 (0.1907)	−0.0027 (0.1894)
	30	0.0453 (0.3061)	−0.0014 (0.2904)	−0.0032 (0.2901)	0.0514 (0.1661)	0.0030 (0.1507)	−0.0001 (0.1502)
	40	0.0343 (0.2611)	−0.0007 (0.2508)	−0.0016 (0.2508)	0.0363 (0.1376)	0.0006 (0.1281)	−0.0012 (0.1280)
	50	0.0277 (0.2316)	−0.0003 (0.2243)	−0.0008 (0.2243)	0.0290 (0.1210)	0.0006 (0.1142)	−0.0005 (0.1142)
1.5	10	0.0774 (0.3748)	0.0041 (0.3418)	−0.0022 (0.3400)	0.2519 (0.5802)	0.0102 (0.4507)	−0.0444 (0.4406)
	20	0.0349 (0.2499)	−0.0018 (0.2386)	−0.0035 (0.2382)	0.1142 (0.3261)	0.0029 (0.2844)	−0.0084 (0.2827)
	30	0.0223 (0.1982)	−0.0023 (0.1922)	−0.0030 (0.1922)	0.0722 (0.2482)	−0.0001 (0.2265)	−0.0048 (0.2260)
	40	0.0165 (0.1712)	−0.0019 (0.1673)	−0.0024 (0.1673)	0.0547 (0.2070)	0.0011 (0.1927)	−0.0015 (0.1925)
	50	0.0145 (0.1531)	−0.0002 (0.1502)	−0.0004 (0.1502)	0.0428 (0.1806)	0.0002 (0.1706)	−0.0015 (0.1705)
2.0	10	0.0509 (0.2769)	0.0039 (0.2586)	0.0003 (0.2576)	0.3437 (0.7829)	0.0204 (0.6067)	−0.0474 (0.6075)
	20	0.0255 (0.1856)	0.0016 (0.1791)	0.0008 (0.1790)	0.1519 (0.4338)	0.0035 (0.3783)	−0.0116 (0.3759)
	30	0.0159 (0.1486)	−0.0001 (0.1452)	−0.0004 (0.1452)	0.0959 (0.3274)	−0.0005 (0.2987)	−0.0069 (0.2979)
	40	0.0119 (0.1271)	−0.0002 (0.1249)	−0.0004 (0.1249)	0.0708 (0.2740)	−0.0006 (0.2555)	−0.0040 (0.2554)
	50	0.0099 (0.1138)	0.0002 (0.1121)	0.0000 (0.1122)	0.0535 (0.2385)	−0.0031 (0.2261)	−0.0053 (0.2260)
3.0	10	0.0300 (0.1814)	0.0036 (0.1734)	0.0016 (0.1730)	0.5026 (1.1656)	0.0194 (0.9068)	−0.0356 (0.9994)
	20	0.0145 (0.1222)	0.0010 (0.1193)	0.0005 (0.1193)	0.2205 (0.6523)	−0.0017 (0.5715)	−0.0245 (0.5670)
	30	0.0105 (0.0973)	0.0014 (0.0956)	0.0012 (0.0957)	0.1356 (0.4866)	−0.0086 (0.4459)	−0.0181 (0.4448)
	40	0.0074 (0.0833)	0.0005 (0.0823)	0.0005 (0.0823)	0.0998 (0.4089)	−0.0071 (0.3829)	−0.0123 (0.3825)
	50	0.0058 (0.0740)	0.0003 (0.0733)	0.0003 (0.0733)	0.0818 (0.3606)	−0.0032 (0.3415)	−0.0065 (0.3417)
5.0	10	0.0176 (0.1061)	0.0037 (0.1025)	0.0011 (0.1027)	0.6409 (1.5625)	−0.1373 (1.2361)	0.0674 (1.5790)
	20	0.0083 (0.0724)	0.0013 (0.0711)	0.0008 (0.0713)	0.3651 (1.0775)	−0.0049 (0.9438)	0.0066 (1.0257)
	30	0.0053 (0.0583)	0.0005 (0.0577)	0.0004 (0.0578)	0.2433 (0.8333)	0.0022 (0.7604)	−0.0061 (0.7766)
	40	0.0040 (0.0504)	0.0005 (0.0499)	0.0004 (0.0500)	0.1759 (0.6844)	−0.0026 (0.6386)	−0.0099 (0.6429)
	50	0.0035 (0.0447)	0.0006 (0.0444)	0.0006 (0.0444)	0.1364 (0.5975)	−0.0053 (0.5657)	−0.0105 (0.5663)

Table 4. Estimated bias (root mean-squared error) for μ and β, ($\mu = 2.0$).

		Estimator of μ			Estimator of β		
β	n	MLE	BCE	PBE	MLE	BCE	PBE
0.5	10	0.5138 (1.7660)	−0.2165 (1.1873)	0.1287 (1.7503)	0.0883 (0.1992)	0.0072 (0.1541)	−0.0132 (0.1505)
	20	0.2967 (1.2102)	−0.0433 (0.9674)	0.0325 (1.1999)	0.0382 (0.1101)	0.0011 (0.0961)	−0.0033 (0.0960)
	30	0.1987 (0.9317)	−0.0182 (0.8014)	−0.0078 (0.8714)	0.0246 (0.0826)	0.0005 (0.0752)	−0.0013 (0.0752)
	40	0.1472 (0.7677)	−0.0116 (0.6869)	−0.0136 (0.7082)	0.0181 (0.0694)	0.0002 (0.0647)	−0.0007 (0.0647)
	50	0.1191 (0.6702)	−0.0062 (0.6127)	−0.0108 (0.6173)	0.0144 (0.0606)	0.0002 (0.0573)	−0.0004 (0.0572)
1.0	10	0.2023 (0.8311)	0.0074 (0.7135)	0.0044 (0.7515)	0.1669 (0.3836)	0.0059 (0.2978)	−0.0312 (0.2896)
	20	0.0905 (0.5229)	−0.0040 (0.4841)	−0.0092 (0.4831)	0.0742 (0.2161)	0.0001 (0.1890)	−0.0075 (0.1877)
	30	0.0583 (0.4119)	−0.0042 (0.3911)	−0.0066 (0.3909)	0.0479 (0.1637)	−0.0003 (0.1494)	−0.0035 (0.1490)
	40	0.0440 (0.3515)	−0.0027 (0.3379)	−0.0040 (0.3376)	0.0348 (0.1371)	−0.0009 (0.1280)	−0.0027 (0.1279)
	50	0.0329 (0.3105)	−0.0044 (0.3011)	−0.0052 (0.3010)	0.0280 (0.1200)	−0.0004 (0.1134)	−0.0015 (0.1134)
1.5	10	0.1036 (0.5053)	0.0064 (0.4614)	−0.0013 (0.4612)	0.2610 (0.5912)	0.0181 (0.4577)	−0.0367 (0.4470)
	20	0.0482 (0.3338)	−0.0007 (0.3182)	−0.0029 (0.3175)	0.1187 (0.3333)	0.0071 (0.2901)	−0.0043 (0.2879)
	30	0.0317 (0.2668)	−0.0010 (0.2584)	−0.0020 (0.2584)	0.0748 (0.2496)	0.0023 (0.2272)	−0.0024 (0.2265)
	40	0.0231 (0.2300)	−0.0014 (0.2247)	−0.0020 (0.2248)	0.0571 (0.2083)	0.0034 (0.1934)	0.0008 (0.1932)
	50	0.0189 (0.2053)	−0.0008 (0.2015)	−0.0010 (0.2014)	0.0443 (0.1797)	0.0017 (0.1694)	0.0000 (0.1693)
2.0	10	0.0755 (0.3740)	0.0124 (0.3489)	0.0076 (0.3473)	0.3334 (0.7797)	0.0115 (0.6077)	−0.0555 (0.6130)
	20	0.0396 (0.2519)	0.0075 (0.2426)	0.0065 (0.2423)	0.1457 (0.4371)	−0.0023 (0.3837)	−0.0172 (0.3817)
	30	0.0247 (0.2002)	0.0033 (0.1953)	0.0028 (0.1952)	0.0966 (0.3338)	0.0002 (0.3048)	−0.0062 (0.3043)
	40	0.0185 (0.1716)	0.0024 (0.1684)	0.0021 (0.1685)	0.0718 (0.2808)	0.0003 (0.2621)	−0.0032 (0.2619)
	50	0.0144 (0.1520)	0.0015 (0.1497)	0.0013 (0.1497)	0.0566 (0.2458)	−0.0002 (0.2327)	−0.0023 (0.2326)
3.0	10	0.0397 (0.2356)	0.0048 (0.2251)	0.0023 (0.2248)	0.5037 (1.1263)	0.0203 (0.8687)	−0.0370 (0.9505)
	20	0.0194 (0.1611)	0.0014 (0.1572)	0.0008 (0.1573)	0.2311 (0.6558)	0.0082 (0.5715)	−0.0141 (0.5692)
	30	0.0129 (0.1297)	0.0008 (0.1277)	0.0006 (0.1277)	0.1504 (0.4934)	0.0055 (0.4483)	−0.0040 (0.4473)
	40	0.0090 (0.1117)	−0.0001 (0.1104)	−0.0002 (0.1104)	0.1120 (0.4103)	0.0047 (0.3811)	−0.0005 (0.3807)
	50	0.0077 (0.1002)	0.0004 (0.0992)	0.0003 (0.0993)	0.0887 (0.3613)	0.0035 (0.3406)	0.0002 (0.3403)
5.0	10	0.0206 (0.1393)	0.0024 (0.1351)	−0.0014 (0.1353)	0.6831 (1.5818)	−0.1009 (1.2340)	0.1104 (1.5859)
	20	0.0099 (0.0969)	0.0006 (0.0953)	−0.0001 (0.0955)	0.3734 (1.0780)	0.0027 (0.9416)	0.0150 (1.0249)
	30	0.0061 (0.0784)	−0.0001 (0.0776)	−0.0004 (0.0776)	0.2461 (0.8283)	0.0049 (0.7545)	−0.0024 (0.7738)
	40	0.0049 (0.0677)	0.0001 (0.0672)	0.0001 (0.0672)	0.1787 (0.6845)	0.0001 (0.6380)	−0.0073 (0.6416)
	50	0.0040 (0.0604)	0.0002 (0.0600)	0.0002 (0.0600)	0.1401 (0.5962)	−0.0017 (0.5635)	−0.0069 (0.5635)

estimates are smaller than those of the uncorrected estimates (see Tables 1–4). Thus, it is clear that the bias corrected estimators (BCE and PBE) also achieve root mean-squared error reduction. Note that all the estimators show the property of consistency i.e., the RMSE decreases as sample size increases. Thus, these simulation results show that second-order bias reduction can be quite successful in bringing the estimates closer to their true values.

5. Illustrative example

In this section, we consider a real life data set and illustrate the methods proposed in the previous sections. The data set is from Bjerkedal (1960), and it represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. Table 5 represents the point estimates of μ and β along with standard errors of MLEs, BCEs and PBEs. It is observed that BCE and PBE estimates provide the lowest standard errors for β whereas for μ, the uncorrected MLEs provides the lowest standard error. We also note that the BCE and PBE

Table 5. MLEs and bias-corrected MLEs (standard error).

Estimators	μ	β
MLE	54.1756 (4.7504)	1.4152 (0.1300)
BCE	53.7707 (4.8070)	1.3880 (0.1275)
PBE	53.7306 (4.8176)	1.3840 (0.1272)

estimates of μ and β are smaller than the uncorrected MLEs. Thus it is clear that the MLEs overestimates μ and β.

6. Concluding remarks

In this paper, we have adopted a "corrective" approach to derive analytical expressions for bias-corrected maximum likelihood estimator for the parameters of the IW distribution. The biases of the proposed estimators are of order $\mathcal{O}(n^{-2})$, whereas for the MLEs they are of order $\mathcal{O}(n^{-1})$, indicating that the proposed estimators converge to their true value considerably faster than those of the MLEs. Besides, we have also considered an alternative bias-correction technique through Efron's Bootstrap resampling. The numerical evidence shows that the proposed bias corrected estimators are quite attractive because they outperform the MLEs in terms of bias and RMSE. We also observe that Bootstrap bias correction is less effective than the analytic correction in terms of bias reduction, and also in terms of root mean squared error (except for some cases). The proposed bias-corrected estimators are strongly recommended over MLE, especially when the sample size is small or moderate, which is often encountered in the context of reliability analysis.

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