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Improved parameter estimation of the Chaudhry and Ahmad distribution with climate applications

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Abstract

The Chaudhry-Ahmad distribution is a two-parameter continuous probability distribution obtained as a solution to a generalized Pearson system of differential equation. Although its probability density curve resembles the inverse-Gaussian, gamma, log-normal, Weibull and other distributions, it has been neglected in the analysis of right-skewed data. The purpose of this paper is three folded. Firstly, to reparametrize the Chaudhry and Ahmad distribution and present some of its basic properties. Secondly to derive the analytical bias-corrected maximum likelihood estimators applying the Cox-Snell methodology and thirdly to study, by MC simulations, the small-sample properties of the maximum likelihood estimators and their bias-corrected versions, obtained from the Cox-Snell formula and by parametric bootstrap technique. The numerical results show, for both parameters, that the maximum likelihood estimators are highly biased, especially in small samples. On the other hand, both, the analytical and bootstrap methodologies, significantly reduce the biases and the mean-squared errors. It is apparent from the results that the analytical bias-correction is more efficient than bootstrap resamples. Finally, wind speed data from six weather stations distributed in the state of Tocantins in Brazil is used to illustrate the applicability of the proposed methods.

Keywords: Bootstrap bias correction · Cox-Snell bias-correction · Maximum likelihood estimation · Monte Carlo simulation · Wind speed data.

Mathematics Subject Classification: Primary 60E05 · Secondary 62F10.

1. Introduction

Chaudhry and Ahmad (1993) introduced a nonnegative two-parameter probability distribution, called the Chaudhry-Ahmad (CA) distribution as a solution of the generalized Pearson system of differential equation. It is noteworthy that from the generalized Pearson system of probability distributions, many continuous probability density functions (PDFs) can be generated (Sankaran et al., 2003; Stavroyiannis, 2014). Indeed, as discussed in Shakil et al. (2010, 2016), the well known families of distributions such as the normal and

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Student-t (known as Pearson type VII), beta distribution (known as Pearson type I) and gamma distribution (known as Pearson type III), introduced by Karl Pearson during the late 19th century (Pearson, 1893, 1895, 1901, 1916), can be generated as a solution to Equation (1) by proper choice of its parameters.

Although the CA PDF curve resembles the inverse-Gaussian (IG), gamma, log-normal, Weibull and other distributions, it has not been widely explored in the statistical literature. Recently, Shakil et al. (2010) derived a family of distribution, which includes the CA distribution as a special case. To the best of our knowledge, there are only two real data analysis considering the CA distribution. Nanos and Montero (2001) showed that CA distribution fitted better than the Weibull distribution in a problem involving prediction of the diameter distribution of a stand. In Nanos et al. (2000) the Weibull and CA distributions were used to model resin production distributions for maritime pine stands.

It is important to point out that the CA distribution is capable of modeling increasing hazard rate funtions (HRFs). There are many situations where only increasing HRFs are used or observed: Woosley and Cossman (2007) observed that drugs during clinical development have increasing HRFs; Tsarouhas and Arvanitoyannis (2010) showed that machines of the bread production display increasing HRFs; Koutras (2011) observed that software degradation times have increasing HRFs; Lai (2013) investigated the optimum number of minimal repairs for systems have increasing hazard rates and so on.

Although the maximum likelihood (ML) estimators have many appealing properties (Edwards, 1992; Lehmann and Casella, 1998), it is also well known that ML estimators could be biased, especially when the study is being done in small samples. Owing to this reason, researchers strive to develop nearly unbiased estimators for the parameters of several probability distributions. Notable among them are Saha and Paul (2005), Lemonte et al. (2007), Giles and Feng (2009), Lagos-Àlvarez et al. (2011), Giles (2012a), Giles (2012b), Schwartz et al. (2013), Giles et al. (2013), Teimouri and Nadarajah (2013), Ling and Giles (2014), Zhang and Liu (2015), Teimouri and Nadarajah (2016), Reath (2016), Schwartz and Giles (2016), Wang and Wang (2017), Mazucheli and Dey (2018), Mazucheli et al. (2018), Mazucheli et al. (2020) and references cited therein.

The objective of this paper is to perform improved parameter estimation of the CA distribution. We consider the analytical methodology introduced by Cox-Snell (1968) and the parametric bootstrap resampling method (Efron, 1982). We describe two corrective approaches to bias-correction, both methods reduce the biases of the ML estimators to the second order magnitude.

After this introduction, the paper is organized as follows. In Section 2, we introduce the CA distribution and deduce expressions used to obtain the ML estimators of its parameters, calculating the expected Fisher information matrix. In Section 3, by using the Cox-Snell formula, we derive analytical expressions for the second order biases of the maximum likelihood estimators, and also discuss the bootstrap bias correction. A Monte Carlo (MC) simulation study is carried out in Section 4 to compare the ML estimators and their bias-corrected versions, obtained from the Cox-Snell formula and parametric bootstrap technique. An application by using wind speed data from Brazil is provided also in this section. As a result of this application, we are able to provide, for example, better estimates of most frequent wind speeds observed at various stations. Some concluding remarks are presented in Section 5.

2. Preliminaries, model description and estimation

In this section, we provide background on the CA distribution and the ML estimators of its parameters, as well as the corresponding expected Fisher information matrix.

2.1 Background on the Chaudhry-Ahmad distribution

Chaudhry and Ahmad (1993) developed a two-parameter probability distribution as a solution of the generalized Pearson system of differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{c_0 + c_1\,x + c_2\,x^2 + \cdots + c_m\,x^m}{c'_0 + c'_1\,x + c'_2\,x^2 + \cdots + c'_n\,x^n}\, f(x), \tag{1}$$

where $m, n \geq 1$ are integers, and the coefficients c and c' are real numbers. These authors considered a special case of Equation (1) taking $m = 4$, $n = 3$, $c'_0 = c'_1 = c'_2 = 0$, $c_4/2\,c'_3 = -2\,\alpha$, $c_0/2\,c'_3 = 2\,\beta$ and $c'_3 \neq 0$. This distribution, which now bear their names, can also be obtained as the root reciprocal of the inverse Gaussian distribution, that is, the distribution of the random variable $X = 1\,/\,\sqrt{Y}$, where $Y \sim \mathrm{IG}(\mu, \lambda)$ with $\mu = (\alpha\,/\,\beta)^{1/2}$ and $\lambda = 2\,\alpha$.

The cumulative distribution function (CDF) of the CA distribution is given by

$$F(x;\alpha,\beta) = \Phi\left[\sqrt{2}\,\left(\sqrt{\alpha}\,x - \sqrt{\beta}\,x^{-1}\right)\right] - \exp\left(4\,\sqrt{\alpha\,\beta}\right)\,\Phi\left[-\sqrt{2}\,\left(\sqrt{\alpha}\,x + \sqrt{\beta}\,x^{-1}\right)\right], \tag{2}$$

where x, α, $\beta > 0$ and Φ denotes the CDF of a standard normal distribution.

Solving the orthogonality differential equation of Cox and Reid (1987), we consider in Equation (2) $\beta = \alpha\lambda^4$, such that λ will be the mode of the PDF. The advantage of such parametrization is that λ has a direct interpretation and it is orthogonal to α. Thus, from Equation (2), the PDF of a CA distributed random variable with parameters α and λ can be written as

$$f(x;\alpha,\lambda) = 2\,\sqrt{\frac{\alpha}{\pi}}\,\exp\left[-\left(\sqrt{\alpha}\,x - \lambda^2\sqrt{\alpha}\,x^{-1}\right)^2\right]. \tag{3}$$

Figure 1 displays the PDF and the HRF curves considering different values of α and $\lambda = 1$ (λ is a location parameter). We observe that the PDF is skewed to the right and unimodal with turning point at $x_{\max} = \lambda = 1$. We also observe that the HRF of CA distribution is monotone increasing.

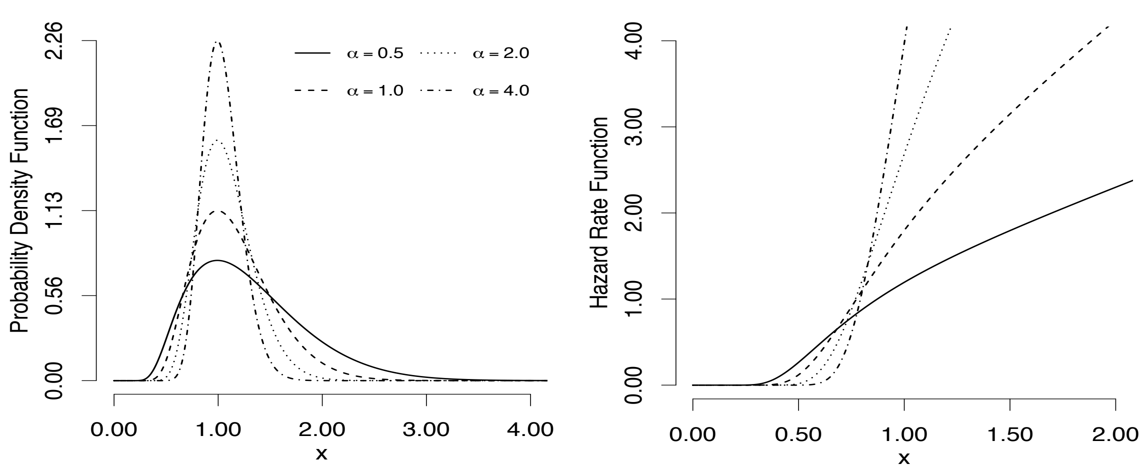

Figure 1. PDF and HRF of the CA distribution for $\alpha = (0.5, 1.0, 2.0$ and $4.0)$ and $\lambda = 1$.

The kth moment about the origin of CA distribution is given by

$$\mu_k' = 2\sqrt{\frac{\alpha}{\pi}}\,\exp\left(2\,\alpha\,\lambda^2\right)\,\lambda^{k+1}\,K_{\frac{k}{2}+\frac{1}{2}}(2\,\alpha\,\lambda^2), \tag{4}$$

where K_ν denotes the modified Bessel function of the second kind (Abramowitz and Stegun, 1974). In particular, from Equation (4), the first four moments about the origin are stated as

$$\begin{aligned}
\mu_1' &= 2\sqrt{\frac{\alpha}{\pi}}\,\exp\left(2\,\alpha\,\lambda^2\right)\,\lambda^2\,K_{\frac{1}{2}+\frac{1}{2}}(2\,\alpha\,\lambda^2),\\
\mu_2' &= \frac{2\,\alpha\,\lambda^2+1}{2\,\alpha},\\
\mu_3' &= 2\sqrt{\frac{\alpha}{\pi}}\,\exp\left(2\,\alpha\,\lambda^2\right)\,\lambda^4\,K_{\frac{3}{2}+\frac{1}{2}}(2\,\alpha\,\lambda^2),\\
\mu_4' &= \frac{4\,\alpha^2\,\lambda^4+6\,\alpha\,\lambda^2+3}{4\,\alpha^4}.
\end{aligned}$$

2.2 Maximum likelihood estimation

Suppose that $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$ is a random sample of size n from CA distribution with PDF given by Equation (3) and $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ its observations. The log-likelihood function for $\boldsymbol{\theta} = (\alpha, \lambda)$ is given by

$$\ell(\boldsymbol{\theta};\boldsymbol{x}) \propto \frac{n}{2}\log(\alpha) + 2\,n\,\alpha\,\lambda^2 - \alpha\sum_{i=1}^{n} x_i^2 - \lambda^4\,\alpha\sum_{i=1}^{n} x_i^{-2}. \tag{5}$$

Differentiating in Equation (5) with respect to α and λ, we have the score vector $\boldsymbol{U}_{\boldsymbol{\theta}} = (U_\alpha, U_\lambda)^\top$ with components given by

$$U_\alpha = \frac{n}{2\,\alpha} + 2\,n\,\lambda^2 - \sum_{i=1}^{n} x_i^2 - \lambda^4\sum_{i=1}^{n} x_i^{-2}, \tag{6}$$

$$U_\lambda = 4\,n\,\alpha\,\lambda - 4\,\lambda^3\alpha\sum_{i=1}^{n} x_i^{-2}. \tag{7}$$

After simple algebraic manipulation of Equations (6) and (7), note that the ML estimates of α and λ can be written as $\widehat{\lambda} = (m_{-2}')^{-1/2}$ and $\widehat{\alpha} = [2(m_{-2}'\,m_2' - 1)\,m_{-2}']^{-1}(m_{-2}')^2$, where $m_2' = (1/n)\sum_{i=1}^{n} x_i^2$ and $m_{-2}' = (1/n)\sum_{i=1}^{n} x_i^{-2}$.

The expected Fisher information matrix of $\boldsymbol{\theta}$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = [I_{ij}] = -n\mathrm{E}\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log(f(x_i;\boldsymbol{\theta}))\right) = \begin{bmatrix} -\dfrac{n}{2\,\alpha^2} & 0 \\ 0 & -8\,n\,\alpha \end{bmatrix}, \quad i,j = 1,2. \tag{8}$$

From Equation (8), we observe that the information matrix is diagonal, which means that the ML estimators are asymptotically independent. Hence, the asymptotic variance of $\widehat{\alpha}$ and $\widehat{\lambda}$ are given, respectively, by

$$\mathrm{Var}(\widehat{\alpha}) = \frac{2\,\alpha^2}{n}, \quad \mathrm{Var}(\widehat{\lambda}) = \frac{1}{8\,n\,\alpha}. \tag{9}$$

The asymptotic variance of $\widehat{\lambda}$ only depends on α. Thus, as α decreases, the variance of $\widehat{\lambda}$ increases. The asymptotic $100(1-\delta)$ confidence intervals for α and λ can be obtained respectively as

$$\widehat{\alpha} \pm z_{\delta/2}\sqrt{\widehat{\mathrm{Var}}(\widehat{\alpha})}, \quad \widehat{\lambda} \pm z_{\delta/2}\sqrt{\widehat{\mathrm{Var}}(\widehat{\lambda})}, \tag{10}$$

where $z_{\delta/2}$ indicated in Equation (10) denotes the $100(1-\delta/2)$ percentile of the standard normal distribution.

3. Bias-corrected maximum likelihood estimators

In this section, we derive analytical expressions for the second order biases of the maximum likelihood estimators by using the Cox-Snell formula, and also discuss the bootstrap bias correction.

3.1 Cox-Snell analytic bias correction

Let $\ell(\boldsymbol{\theta}; \boldsymbol{x})$ denote the log-likelihood function of a p-dimensional parameter vector $\boldsymbol{\theta}$ based on a sample of observations $\boldsymbol{x}$. We assume the following regularity conditions on the behavior of the log-likelihood function (Cox and Hinkley, 1979):

(a) X_i, for $i = 1, \ldots, n$, are independent and identically distributed random variables.
(b) The parameter space of $\boldsymbol{\theta}$ is compact.
(c) The true but unknown parameter value $\boldsymbol{\theta}_0$ is identified, that is,

$$\boldsymbol{\theta}_0 = \arg\max_{\boldsymbol{\theta}} \mathrm{E}_{\boldsymbol{\theta}_0}\left[\log\left(f\left(x_i; \boldsymbol{\theta}\right)\right)\right].$$

(d) The likelihood function

$$\ell\left(\boldsymbol{\theta}; \boldsymbol{x}\right) = \sum_{i=1}^{n} \log\left(f\left(x_i; \boldsymbol{\theta}\right)\right)$$

is continuous in $\boldsymbol{\theta}$.
(e) $\mathrm{E}_{\boldsymbol{\theta}_0}\left[\log\left(f\left(x_i; \boldsymbol{\theta}\right)\right)\right]$ exists.
(f) The log-likelihood function is such that $(1/n)\ell(\boldsymbol{\theta}; \boldsymbol{x})$ converges almost surely (in probability) to $\mathrm{E}_{\boldsymbol{\theta}_0}[\log(f(x_i; \boldsymbol{\theta}))]$ uniformly in $\boldsymbol{\theta}$.

Conditions (a) to (d) are clearly satisfied for the CA distribution. Conditions (e) and (f) are also satisfied since, for all $\alpha > 0$ and $\beta > 0$,

$$\int_0^\infty x^2 \exp\left(-\alpha x^2 - \frac{\lambda^4 \alpha}{x^2}\right) \mathrm{d}x < \infty,$$
$$\int_0^\infty x^{-2} \exp\left(-\alpha x^2 - \frac{\lambda^4 \alpha}{x^2}\right) \mathrm{d}x < \infty.$$

The joint cumulants of the derivatives of ℓ are given by

$$\boldsymbol{I}_{ij} = \mathrm{E}\left[\frac{\partial^2 \ell}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j}\right], \quad \boldsymbol{I}_{ijl} = \mathrm{E}\left[\frac{\partial^3 \ell}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j \, \partial \boldsymbol{\theta}_l}\right], \quad \boldsymbol{I}_{ij,l} = \mathrm{E}\left[\left(\frac{\partial^2 \ell}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j}\right)\left(\frac{\partial \ell}{\partial \boldsymbol{\theta}_l}\right)\right],$$

for $i, j, l = 1, \ldots, p$. All these expression are assumed to be of order $\mathcal{O}(n)$.

Cox-Snell (1968) showed that when the samples are independent, but not necessarily identically distributed, the bias of the rth element of the ML estimator of $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}$, can be expressed as

$$\mathcal{B}(\widehat{\boldsymbol{\theta}}_r) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \boldsymbol{I}^{ri} \, \boldsymbol{I}^{jl} \, [0.5 \boldsymbol{I}_{ijl} + \boldsymbol{I}_{ij,l}] + \mathcal{O}\left(n^{-2}\right), \tag{11}$$

where $r = 1, \ldots, p$ and $\boldsymbol{I}^{ij}$ denotes the (i, j)th element of the inverse of the expected Fisher information matrix.

In respect to the orthogonally parametrization of CA distribution, after extensive algebra, it can be shown that $\boldsymbol{I}_{111} = -24 \, n \, \alpha / \lambda$, $\boldsymbol{I}_{122} = \boldsymbol{I}_{212} = \boldsymbol{I}_{221} = -8 \, n$, $\boldsymbol{I}_{222} = n / \alpha^3$, $\boldsymbol{I}_{11,1} = 24 \, n \, \alpha / \lambda$, $\boldsymbol{I}_{12,2} = \boldsymbol{I}_{21,2} = 8 \, n$ and all other terms are equal to zero. Hence, the second-order bias of the ML estimators of α and λ are given respectively by

$$\mathcal{B}(\widehat{\alpha}) = \frac{3 \, \alpha}{n} \tag{12}$$

and

$$\mathcal{B}(\widehat{\lambda}) = \frac{3}{16 \, n \, \alpha \, \lambda}, \tag{13}$$

Using Equations (12) and (13), we define the bias-corrected (BC) estimator as

$$\widehat{\alpha}_{\mathrm{BC}} = \widehat{\alpha} - \widehat{\mathcal{B}}(\widehat{\alpha}), \quad \widehat{\lambda}_{\mathrm{BC}} = \widehat{\lambda} - \widehat{\mathcal{B}}(\widehat{\lambda}). \tag{14}$$

Note that $\widehat{\alpha}_{\mathrm{BC}}$ and $\widehat{\lambda}_{\mathrm{BC}}$ defined in Equation (14) have bias of order $\mathcal{O}(n^{-2})$ as indicated in (11). Thus, it is expected that they have superior sampling properties relative to $\widehat{\alpha}$ and $\widehat{\lambda}$. We also empathize that the bias-corrected ML estimator for β in the original parametrization Equation (2) is obtained from $\widehat{\beta} - (3\sqrt{\widehat{\alpha}} \, \widehat{\beta} + 1.5 \sqrt{\widehat{\beta}}) / \sqrt{\widehat{\alpha}} n$.

3.2 Parametric bootstrap bias correction

An alternative approach to analytically bias-corrected ML estimators is based on bootstrap resampling scheme (Efron and Tibshirani, 1993; Davison and Hinkley, 1997). In this method the bias correction is performed numerically without deriving analytical expression for the bias function. In fact, the parametric bootstrap bias correction (PB) estimates use the ML estimates of the data to generate pseudo-random samples from the distribution to estimate the bias and then subtract the bias from the ML estimates.

Let $\widehat{\boldsymbol{\theta}}_{(\cdot)}$ be the average value of the ML estimator from B bootstrap replications, based on a pseudo-sample of size n generated from Equation (3) using the parameters of the ML estimates $\widehat{\boldsymbol{\theta}}$. The estimated bias of $\widehat{\boldsymbol{\theta}}$ is defined as $\widehat{\mathcal{B}}(\widehat{\boldsymbol{\theta}}) = \widehat{\boldsymbol{\theta}}_{(\cdot)} - \widehat{\boldsymbol{\theta}}$. Then, the bootstrap bias-corrected estimator is $\widehat{\boldsymbol{\theta}}_{\mathrm{PB}} = 2 \, \widehat{\boldsymbol{\theta}} - \widehat{\boldsymbol{\theta}}_{(\cdot)}$.

4. Numerical evaluations

In this section, we carry out a MC simulation study to compare the ML estimators and their bias-corrected versions. In addition, we illustrate the applicability of the CA distribution for bias corrections to the wind speed data.

4.1 Simulation study

Our MC simulation study is conducted to compare the finite-sample behavior of the ML estimators and their bias-corrections obtained by Cox-Snell methodology (BC) and parametric bootstrap scheme (PB) for the parameters that index the CA distribution. For this purpose, we generate samples of size $n = 10, 20, 30, 40$ and 50 from Equation (3) considering $\alpha = 0.5, 1.0, 1.5, 2.0$ and 4.0 and fixed $\lambda = 1$, since it is a location parameter and the estimators are scale invariant. The behavior of PDF and HRF for these parameters values were illustrated in Figure 1. It is important to note that the mean, variance, skewness and kurtosis of a CA distributed random variable decrease as α increases.

To simulate random variables from a CA distribution, we generated samples from a random variable Y with inverse Gaussian distribution and we used the transformation $X = 1/\sqrt{Y}$.

To assess the performance of the methods under consideration, we calculated the bias and root mean-squared error (RMSE). The number of MC simulations was fixed at $M = 10,000$ and $B = 1,000$ bootstrap replicates were used. All simulations were carried out in Ox Console which is a matrix programming language with object-oriented support developed by Jurgen Doornik (Doornik, 2007).

Table 1 depicts the estimated bias and root mean-squared error, in parentheses, for different values of α and $\lambda = 1$. We can observe that all the estimates show the property of consistency, that is, the RMSEs decrease as sample size increases. We also note that the ML estimates of α are highly biased, particularly when the sample size is small. For instance, the biases of the ML estimates of α for $(n, \alpha) = (10, 0.5)$ and $(n, \alpha) = (10, 4)$ are approximately 22% and 169%, respectively. Also the biases of the ML estimates of α for $(n, \alpha) = (20, 0.5)$ and $(n, \alpha) = (20, 4)$ are approximately 9% and 68%, respectively. The estimates $\widehat{\alpha}_{\text{BC}}$ and $\widehat{\alpha}_{\text{PB}}$ clearly outperform the ML estimates as far as the bias goes. For example, the biases of the BC estimates of α for $(n, \alpha) = (10, 0.5)$ and $(n, \alpha) = (10, 4)$ are approximately 0.3% and 1.5%, respectively. The biases of the PB estimates of α for $(n, \alpha) = (10, 0.5)$ and $(n, \alpha) = (10, 4)$ are approximately 8.9% and 74.6%, respectively. Thus, the proposed estimators achieve substantial bias reduction, especially for the small and moderate sample sizes and therefore, we consider them as better alternatives of the ML estimates of α. We also observe that the bias-corrected estimates are closer to the true parameter values than the unadjusted estimates as sample size increases. Additionally, the estimated root mean-squared errors for α of the bias corrected estimates are smaller than those of the uncorrected estimates. On the other hand, the RMSE of λ are very similar for all estimators.

Now, in order to evaluate the overall performance of each estimation method with respect to the bias and root mean squared error, for each value of n, we use two measures introduced by Cribari-Neto and Vasconcellos (2002). The authors called these quantities as integrated bias squared norm and average root mean squared error. They are calculated as follows

$$\text{IBSQ}_{(k)} = \sqrt{\frac{1}{16}\sum_{h=1}^{16}(r_{h,k})^2}, \quad \text{ARMSE}_{(k)} = \frac{1}{16}\sum_{h=1}^{16}\text{RMSE}_{h,k},$$

Table 1. Estimated bias (root mean-squared error) for α and λ, ($\lambda = 1.0$).

α	n	Estimator of α ML	BC	PB	Estimator of λ ML	BC	PB
	10	0.2191 (0.5098)	0.0034 (0.3223)	-0.0891 (0.2778)	0.0351 (0.1646)	0.0018 (0.1667)	0.0029 (0.1662)
	20	0.0908 (0.2345)	0.0022 (0.1838)	-0.0135 (0.1787)	0.0173 (0.1142)	-0.0003 (0.1149)	-0.0000 (0.1149)
0.5	30	0.0571 (0.1686)	0.0014 (0.1428)	-0.0048 (0.1411)	0.0111 (0.0924)	-0.0009 (0.0929)	-0.0008 (0.0929)
	40	0.0410 (0.1369)	0.0004 (0.1208)	-0.0029 (0.1200)	0.0082 (0.0798)	-0.0009 (0.0801)	-0.0008 (0.0801)
	50	0.0324 (0.1174)	0.0005 (0.1061)	-0.0016 (0.1057)	0.0063 (0.0715)	-0.0010 (0.0717)	-0.0010 (0.0717)
	10	0.4330 (0.9830)	0.0031 (0.6178)	-0.1810 (0.5368)	0.0171 (0.1131)	0.0004 (0.1138)	0.0006 (0.1137)
	20	0.1775 (0.4620)	0.0009 (0.3625)	-0.0303 (0.3526)	0.0085 (0.0793)	-0.0004 (0.0795)	-0.0003 (0.0795)
1.0	30	0.1149 (0.3370)	0.0034 (0.2851)	-0.0090 (0.2818)	0.0059 (0.0648)	-0.0001 (0.0649)	-0.0001 (0.0650)
	40	0.0844 (0.2722)	0.0031 (0.2394)	-0.0035 (0.2380)	0.0046 (0.0559)	0.0000 (0.0560)	0.0000 (0.0560)
	50	0.0667 (0.2334)	0.0027 (0.2103)	-0.0015 (0.2094)	0.0034 (0.0501)	-0.0003 (0.0501)	-0.0003 (0.0501)
	10	0.6543 (1.5198)	0.0080 (0.9603)	-0.2688 (0.8293)	0.0121 (0.0927)	0.0010 (0.0931)	0.0011 (0.0930)
	20	0.2657 (0.6906)	0.0008 (0.5419)	-0.0459 (0.5272)	0.0064 (0.0653)	0.0005 (0.0654)	0.0005 (0.0654)
1.5	30	0.1653 (0.4970)	-0.0013 (0.4218)	-0.0197 (0.4173)	0.0040 (0.0530)	-0.0000 (0.0530)	-0.0000 (0.0530)
	40	0.1213 (0.4035)	-0.0003 (0.3559)	-0.0100 (0.3539)	0.0027 (0.0457)	-0.0004 (0.0458)	-0.0003 (0.0458)
	50	0.0963 (0.3489)	0.0005 (0.3152)	-0.0056 (0.3140)	0.0022 (0.0410)	-0.0002 (0.0410)	-0.0002 (0.0410)
	10	0.8784 (2.0228)	0.0149 (1.2756)	-0.3550 (1.1005)	0.0090 (0.0796)	0.0007 (0.0798)	0.0007 (0.0798)
	20	0.3588 (0.9360)	0.0050 (0.7348)	-0.0575 (0.7144)	0.0042 (0.0559)	-0.0003 (0.0560)	-0.0003 (0.0560)
2.0	30	0.2319 (0.6808)	0.0087 (0.5762)	-0.0161 (0.5694)	0.0026 (0.0456)	-0.0004 (0.0457)	-0.0004 (0.0457)
	40	0.1707 (0.5483)	0.0079 (0.4820)	-0.0054 (0.4788)	0.0020 (0.0397)	-0.0003 (0.0397)	-0.0003 (0.0397)
	50	0.1317 (0.4702)	0.0038 (0.4244)	-0.0044 (0.4227)	0.0015 (0.0354)	-0.0003 (0.0354)	-0.0003 (0.0354)
	10	1.6928 (3.8947)	-0.0150 (2.4553)	-0.7463 (2.1403)	0.0053 (0.0565)	0.0011 (0.0566)	0.0011 (0.0566)
	20	0.6965 (1.8473)	-0.0080 (1.4543)	-0.1329 (1.4157)	0.0027 (0.0398)	0.0004 (0.0398)	0.0004 (0.0398)
4.0	30	0.4358 (1.3260)	-0.0078 (1.1271)	-0.0573 (1.1144)	0.0018 (0.0325)	0.0003 (0.0325)	0.0003 (0.0325)
	40	0.3110 (1.0752)	-0.0123 (0.9521)	-0.0385 (0.9465)	0.0014 (0.0282)	0.0003 (0.0282)	0.0003 (0.0282)
	50	0.2406 (0.9169)	-0.0138 (0.8318)	-0.0301 (0.8289)	0.0011 (0.0253)	0.0002 (0.0253)	0.0002 (0.0253)

where $r_{h,k}$ and $\text{RMSE}_{h,k}$ correspond to the 16 different values of the bias and the root mean squared errors of each estimator given in Table 1. The results are reported in Tables 2-3.

From Table 2, we see that integrated bias squared norm of the corrected estimates (BC and PB) are smaller than ML estimates for both parameter α and λ. From Table 2, we can see that the average root mean-squared error of the corrected estimates (BC and PB) are smaller than ML estimates for α, while for λ the ARMSE are quite similar. Therefore, these simulation results show that second-order bias reduction is quite successful in bringing the corrected estimates closer to their true values.

Table 2. Integrated bias squared norm.

n	Estimator of α ML	BC	PB	Estimator of λ ML	BC	PB
10	0.9274	0.0103	0.3990	0.0189	0.0011	0.0015
20	0.3806	0.0043	0.0695	0.0094	0.0004	0.0004
30	0.2398	0.0055	0.0284	0.0061	0.0005	0.0004
40	0.1728	0.0067	0.0181	0.0045	0.0005	0.0004
50	0.1342	0.0065	0.0139	0.0035	0.0005	0.0005

Table 3. Average root mean-squared error.

n	Estimator of α ML	BC	PB	Estimator of λ ML	BC	PB
10	2.1352	1.3464	1.1701	0.1077	0.1086	0.1084
20	1.0034	0.7892	0.7680	0.0752	0.0755	0.0755
30	0.7226	0.6135	0.6066	0.0611	0.0613	0.0613
40	0.5852	0.5172	0.5141	0.0528	0.0530	0.0530
50	0.5004	0.4532	0.4515	0.0473	0.0474	0.0474

4.2 Wind real data modeling

The data consist of annual maximum wind speed of six weather stations localized in state of Tocantins, Brazil. The data were obtained from the website http://www.inmet.gov.br/portal. Some descriptive statistics of the observed annual maximum wind speed for the stations are summarized in Table 4. Note that the values of skewness are positive for four stations, indicating that the data are right-skewed.

Table 4. Descriptive statistics of the wind speed data for all weather stations.

Station	Period	n	Min	Mean	Med	Max	SD	Skewn	Kurt
82659	1980-2016	34	1.4667	3.0441	3.0667	5.0000	1.1491	0.1541	1.5672
82863	1977-2016	40	2.5667	4.8862	4.6667	8.3333	1.6084	0.4240	2.0874
83033	1993-2016	21	3.6667	5.4817	5.4667	7.1000	1.0460	-0.1572	2.1114
83064	1961-2016	56	2.2667	3.9939	3.9750	6.0000	0.9736	0.1973	2.3523
83228	1975-2016	42	3.1000	4.3657	4.3333	5.6667	0.6096	-0.1140	2.6386
83235	1961-2016	56	2.3333	3.9994	3.6667	6.6667	0.9424	0.8900	3.4780

In Table 5 we report the ML estimates and the bias corrections estimates along with the asymptotic standard errors calculated from Equation (9). We can observe that the maximum likelihood estimates of α and λ are greater than the second order bias corrected estimates for all stations, this suggests that the ML estimates are overestimating the true value of the parameters. We also observe that the corrected ML estimates of α have smaller standard errors than the uncorrected estimates.

In order to test whether the data sets fits the CA distribution and whether the bias-corrected estimates yield better fits than the uncorrected estimates, we perform the goodness-of-fit tests based on Kolmogorov-Smirnov (KS), Cramér-von-Mises (CM) and Anderson-Darling (AD) statistics. The p-values of these statistics are shown in Table 8. We have used the function *mledist* from *fitdistrplus* library, (Delignette-Muller, 2015), available in *R* environment, (R Core Team, 2017), to find the ML estimates. The p-values associated with Kolmogorov-Smirnov, Anderson-Darling and Cramér-von Mises tests were calculated using 10,000 nonparametric bootstrap resamples applying the functions *ks.test*, *ad.test* and *cvm.test*, available in *goftest* (Faraway et al., 2014) R library. Here we are to note that CA distribution can be used to model the annual maximum wind speed data for the six stations. We can see from Table 8 that the p-values of KS, CM and AD computed from bias-corrected estimators are greater than the uncorrected estimator (except KS for one station). This means that bias corrected estimates provide better fits than the ML estimates. This conclusion is also supported by the empirical and fitted CDF plots in Figure 2. Furthermore, in Table 6, we compare the suitability of the CA distribution against eight commonly used probability distributions to modeling wind speed data. The assessment of the goodness-of-fit is based on the log-likelihood values, since all distributions have the same number of parameters. The results are reported in Table 6 and we can see that CA distribution is the best model among the others. The superscripts indicates the rank obtained by the estimation method (the smaller the better). The line named as rank total (TR) shows the sum of the ranks.

Table 5. Point estimates (standard errors) for all weather stations.

		Estimator of α			Estimator of λ	
Station	ML	BC	PB	ML	BC	PB
82659	0.1081 (0.0262)	0.0986 (0.0239)	0.0978 (0.0237)	2.4341 (0.1844)	2.4132 (0.1931)	2.4121 (0.1939)
82863	0.0559 (0.0125)	0.0517 (0.0116)	0.0515 (0.0115)	4.1773 (0.2365)	4.1572 (0.2459)	4.1627 (0.2464)
83033	0.1148 (0.0354)	0.0984 (0.0304)	0.0962 (0.0297)	5.1707 (0.2277)	5.1556 (0.2459)	5.1539 (0.2487)
83064	0.1359 (0.0257)	0.1286 (0.0243)	0.1279 (0.0242)	3.6335 (0.1282)	3.6268 (0.1317)	3.6250 (0.1321)
83228	0.3323 (0.0725)	0.3086 (0.0673)	0.3075 (0.0671)	4.2329 (0.0946)	4.2297 (0.0982)	4.2315 (0.0984)
83235	0.1639 (0.0310)	0.1551 (0.0293)	0.1540 (0.0291)	3.7170 (0.1167)	3.7115 (0.1200)	3.7098 (0.1204)

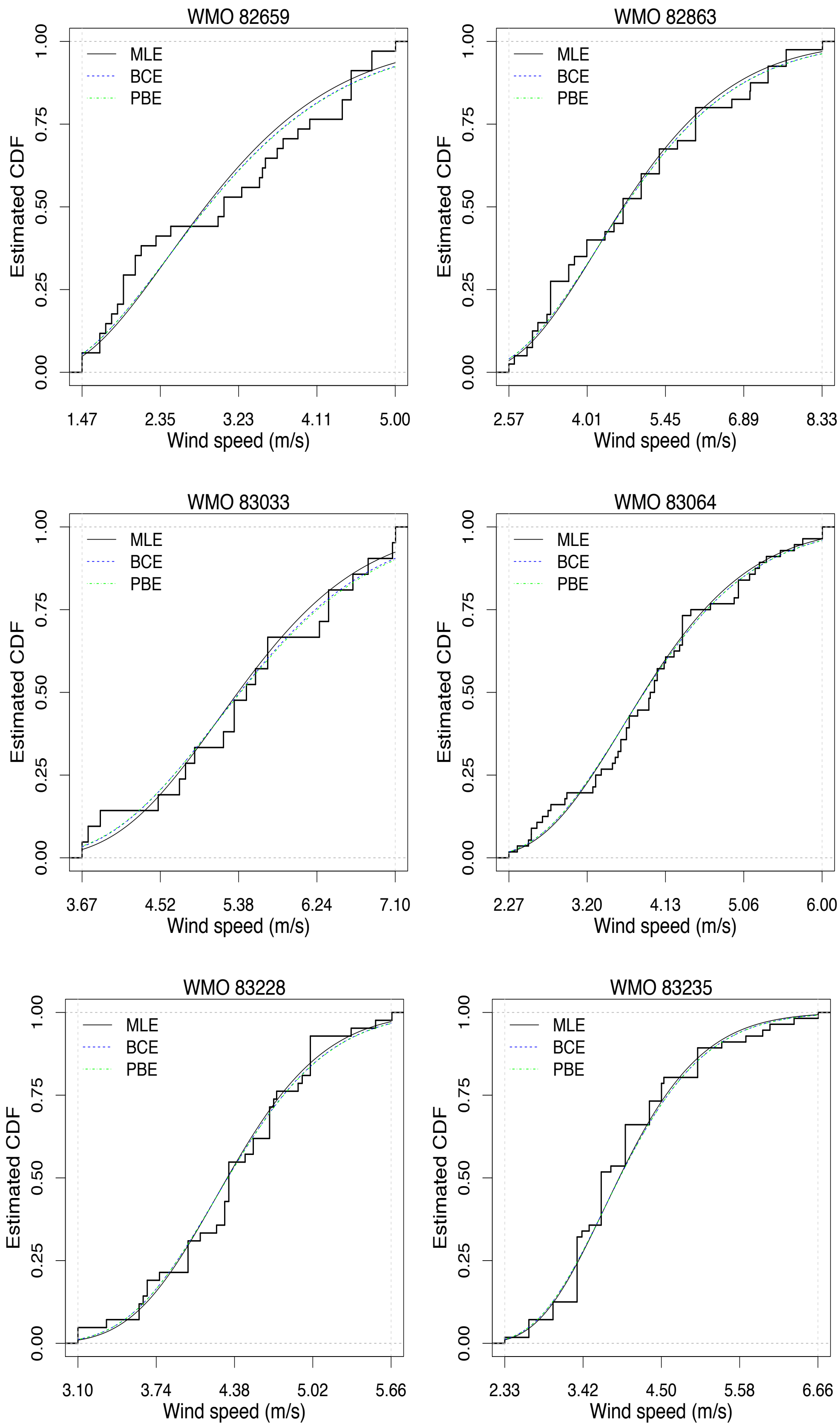

Figure 2. Empirical and fitted CDFs for all examined stations.

Table 6. Negative of the log likelihood values, $-2\log(L)$, of the competing distributions.

Station	CA	Weibull	Gamma	Log-normal	Log-logistic	IG	Gumbel	BS	Nakagami
82659	101.4117^{1}	103.0848^{5}	103.1692^{6}	103.6138^{7}	107.4095^{9}	103.0738^{4}	104.3379^{8}	102.9892^{2}	103.0680^{3}
82863	145.7202^{1}	149.6565^{8}	147.2638^{5}	146.9742^{4}	150.4022^{9}	146.5940^{3}	147.4252^{6}	146.5772^{2}	148.0746^{7}
83033	61.3816^{4}	60.0757^{1}	61.0546^{3}	61.6314^{7}	62.5727^{8}	61.6019^{6}	63.3180^{9}	61.5889^{5}	60.6320^{2}
83064	154.2422^{3}	155.7540^{7}	154.1821^{2}	155.1365^{6}	157.5565^{9}	155.0057^{5}	157.1036^{8}	154.9604^{4}	153.9516^{1}
83228	78.1260^{4}	77.5426^{3}	77.4511^{2}	78.2403^{7}	79.0705^{8}	78.2372^{6}	83.3099^{9}	78.2280^{5}	76.8940^{1}
83235	143.7640^{5}	156.2722^{9}	145.2559^{7}	143.5298^{4}	144.2417^{6}	143.4797^{2}	142.3888^{1}	143.5114^{3}	147.7691^{8}
RT	18^{1}	33^{6}	25^{4}	35^{7}	49^{9}	26^{5}	41^{8}	21^{2}	22^{3}

Table 7. Voung test (p-values) comparing CA distribution with others.

Weibull	Gamma	Log-normal	Log-logistic	IG	Gumbel	BS	Nakagami
0.595 (0.276)	2.146 (0.016)	3.782 (0.000)	8.895 (0.000)	2.538 (0.006)	5.876 (0.000)	2.773 (0.003)	0.882 (0.189)
1.522 (0.064)	2.171 (0.015)	2.460 (0.007)	4.628 (0.000)	1.806 (0.035)	2.156 (0.016)	1.959 (0.025)	1.479 (0.070)
-0.650 (0.742)	-0.788 (0.785)	1.378 (0.084)	1.218 (0.112)	1.564 (0.059)	2.545 (0.005)	1.482 (0.069)	-1.014 (0.845)
0.497 (0.310)	-0.072 (0.529)	2.107 (0.018)	1.929 (0.027)	2.267 (0.012)	3.379 (0.000)	2.203 (0.014)	-0.192 (0.576)
-0.168 (0.567)	-1.282 (0.900)	0.872 (0.192)	0.654 (0.256)	1.090 (0.138)	2.854 (0.002)	1.008 (0.157)	-1.233 (0.891)
3.457 (0.000)	1.943 (0.026)	-0.555 (0.711)	0.244 (0.404)	-0.799 (0.788)	-0.686 (0.754)	-0.750 (0.773)	2.375 (0.009)

Table 8. p-values associated to goodness-of-fit measures for all weather stations.

		KS			CM			AD	
Station	ML	BC	PB	ML	BC	PB	ML	BC	PB
82659	0.4697	0.5185	0.5205	0.3129	0.3970	0.4039	0.3247	0.4056	0.4112
82863	0.6574	0.7285	0.7192	0.7725	0.8539	0.8515	0.7825	0.8570	0.8547
83033	0.9390	0.9763	0.9775	0.8936	0.9252	0.9262	0.8599	0.9042	0.9047
83064	0.8883	0.8736	0.8684	0.7438	0.7689	0.7677	0.7557	0.7962	0.7976
83228	0.4790	0.4961	0.5090	0.6607	0.6748	0.6836	0.7368	0.7534	0.7578
83235	0.2996	0.3054	0.3097	0.4005	0.3801	0.3795	0.4824	0.4735	0.4731

Therefore, using the interpretation for λ given in Section 1, the estimates given in Table 5 can be interpreted as follows. The most frequent wind speed at: station 82659 is around 2.4; station 82863 is around 4.2; station 83033 is around 5.2; station 83064 is around 3.6; station 83228 is around 4.2; station 83235 is around 3.7.

5. Conclusions

In this paper, we have adopted a corrective approach to derive analytical expressions for the second order biases of the maximum likelihood estimators of the parameters of the Chaudhry-Ahmad distribution. Furthermore, we have also considered an alternative bias-correction mechanism through bootstrap resampling. The biases of the proposed estimators are of order $\mathcal{O}\left(n^{-2}\right)$, whereas for the maximum likelihood estimators they are of order $\mathcal{O}\left(n^{-1}\right)$, indicating that the proposed estimates converge to their true value considerably faster than those of the maximum likelihood estimates.

The numerical evidence shows that the proposed bias corrected estimators are quite attractive because they outperform the maximum likelihood estimates in terms of biases, integrated bias squared norm and root mean-squared error. Further, our analytic bias correction is found to be superior to the alternative of bias-correction via the bootstrap in terms of bias reduction. The proposed bias-corrected estimators are strongly recommended over maximum likelihood estimator, especially when the sample size is small or moderate since it has smaller bias and root mean-squared error

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