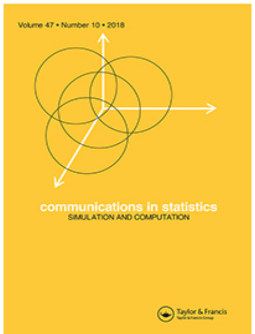

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Improved maximum likelihood estimation of the parameters of the Gamma-Uniform distribution with bias-corrections

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ABSTRACT

A two-parameter Gamma-Uniform distribution was recently introduced as a prominent alternative in modeling bounded phenomena. Unfortunately, however, its maximum likelihood estimators (MLEs) are found to be highly biased in finite samples, a limitation that might effect this model's application in data modeling. In this article, we construct nearly unbiased estimators for the unknown parameters of this distribution by deriving analytical bias-corrected maximum likelihood estimators applying the Cox and Snell methodology, the Firth's method and also via the parametric Bootstrap bias correction approach. Our extensive simulation clearly revealed that the three bias reduction methods yield very good estimates which are nearly unbiased and exhibit comparable efficiency. Finally, we consider a real data set where the variable under enquiry is the proportion of unemployed labor force reported across some 158 nations in 2018 to show case the positive gain of incorporating the bias correction in the model fitting.

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Bootstrap bias-correction; Cox-Snell bias-correction; Firth's method; Gamma-Uniform distribution; Maximum likelihood estimators; Monte Carlo simulation

1. Introduction

Torabi and Hedesh (2012) introduced a new bounded distribution on the interval (a, b) Z(HTML translation failed) which the probability density function (p.d.f.) is defined as

$$f(x \mid \alpha, \beta) = \frac{(b-a)}{\Gamma(\alpha)\ \beta^{\alpha}\ (b-x)^2}\ \exp\left[-\frac{x-a}{\beta\ (b-x)}\right]\left(\frac{x-a}{b-x}\right)^{\alpha-1}, \qquad a < x < b.$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters. The authors named this distribution as Gamma-Uniform (GU), since it arises from the Gamma-Generated class of distributions.

Without loss of generality, we will consider that $a = 0$ and $b = 1$. In this case, we have the following p.d.f.

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\ \beta^{\alpha}\ (1-x)^2}\ \exp\left[-\frac{x}{\beta\ (1-x)}\right]\left(\frac{x}{1-x}\right)^{\alpha-1}, \qquad 0 < x < 1. \tag{1}$$

It should be noted that if Y follows a Gamma distribution with shape (α) and scale (β) parameters, then the random variable $X = Y/(1+Y)$ has p.d.f. given by (1). The cumulative distribution function of X can be written as

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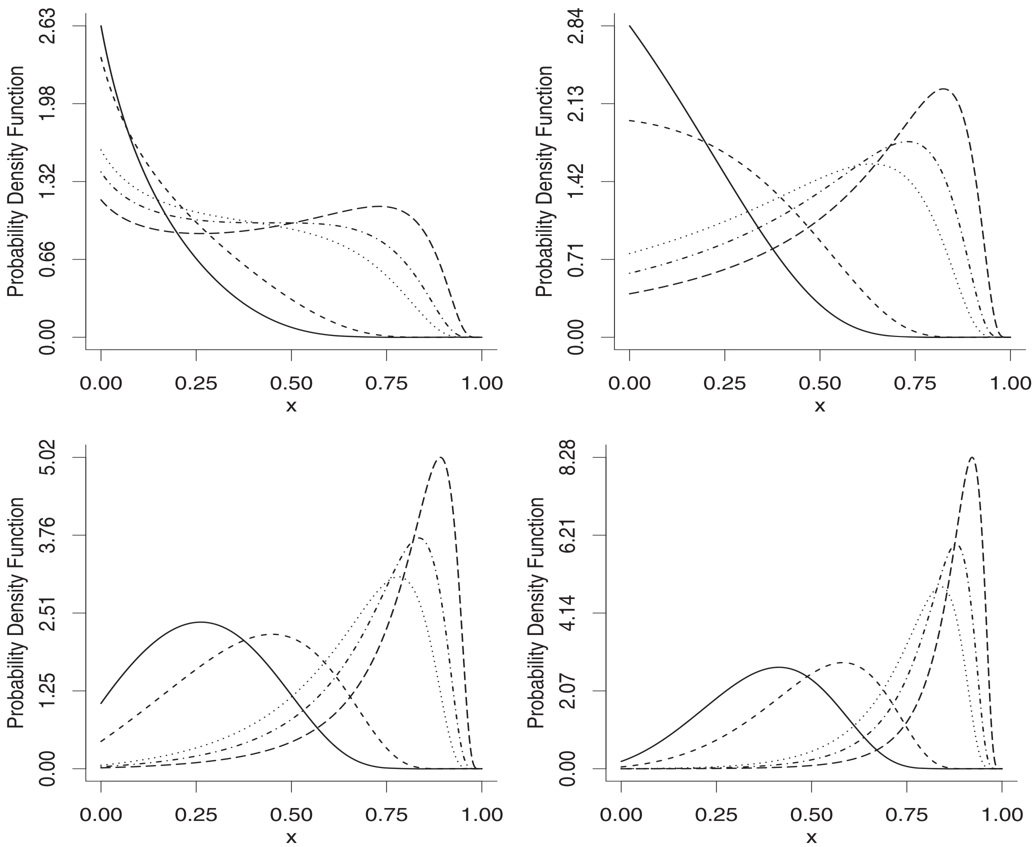

Figure 1. Probability density function of the Gamma-Uniform distribution for selected values of α and β. (left upper panel: $\alpha = 0.5$, right upper panel: $\alpha = 1.0$, left lower panel: $\alpha = 2.0$ and right lower panel: $\alpha = 3.0$; solid line: $\beta = 0.3$, dashed line: $\beta = 0.5$, dotted line: $\beta = 1.5$, dotdash line: $\beta = 2.0$ and longdash line: $\beta = 3.0$).

$$F(x \mid \alpha, \beta) = 1 - Q\left(\alpha, \frac{x}{\beta\ (1-x)}\right) \tag{2}$$

where $Q(a,z) = \frac{1}{\Gamma(a,z)}\Gamma(a)$ is the regularized incomplete gamma function and $\Gamma(a,z) = \int_z^\infty\ t^{a-1}\ \mathrm{e}^{-t}\ \mathrm{d}t$ is the upper incomplete gamma function. In Figure 1 we observe the behavior of the p.d.f. of the Gamma–Uniform distribution for selected values of α and β.

The choice of estimation methodology is important when estimating parameters from any probability distribution. Among all the classical estimation methods, the most frequently used one is the maximum likelihood estimation method (Pawitan 2001). Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. However, it is well known that with finite samples the maximum likelihood estimator (MLE) do not possess any desirable sampling properties. In particular, the MLE is often biased for smaller samples size. Therefore, it is important to derive closed-form expressions for the first-order biases of estimators in order to evaluate the accuracy and also to define estimators with smaller biases which can be used in practical applications for some classes of models. In the statistical literature, several researchers have strived to develop nearly unbiased estimators for the parameters of different distributions. Interested readers may refer to Cribari-Neto and Vasconcellos (2002), Saha and Paul (2005), Lemonte, Cribari-Neto, and Vasconcellos (2007), Giles (2012), Schwartz, Godwin, and Giles (2013), Giles, Feng, and Godwin (2013), Ling and Giles (2014), Schwartz and Giles (2016), Wang and Wang (2017), Mazucheli, Menezes, and

Dey (2018), Mazucheli, Menezes, and Nadarajah (2017), Reath, Dong, and Wang (2018), and Mazucheli and Dey (2018).

The chief goal of this article is to obtain modified MLEs for the parameters of GU distribution that are nearly unbiased. To achieve this, we adopt approaches namely, the methodology introduced by Cox and Snell (1968) which yields analytical expressions for the bias of order $\mathcal{O}(n^{-1})$ and the bias-corrected MLEs through parametric Bootstrap re-sampling method of Efron (1982). We also considered the preventive method introduced by Firth (1993) which consists of transforming the score function.

We assessed the comparative performance of the proposed estimators by Monte Carlo simulations and a real data application. For the application, we have considered a very relevant data set for model fitting which is about the proportion of unemployed labor force in the age group 15–24 across 158 countries. This is an important indicator to understand a country's progress report on achieving the sustainable economic growth. The data is available in the world bank site. The variable here being a proportion suits us to check the GU model. Our findings are encouraging with reduced standard error for the parameter estimators and substantial gain in the p-value of the selection criteria.

The remainder of this article is organized as follows. In the next section, we briefly summarize the maximum likelihood estimators of the two parameters of interest. Section 3 discusses three different approaches to bias correction. Monte Carlo simulations are presented and examined in Sec. 4. In Sec. 5, a real application is presented for illustrative purposes. Finally, Sec. 6 closes the article with a few remarks.

2. Maximum Likelihood estimation

Let $\boldsymbol{x} = (x_1, ..., x_n)$ denote a random sample of size n from the GU distribution with p.d.f. defined in (1). The log-likelihood function for $\boldsymbol{\theta} = (\alpha, \beta),$ apart constant terms, can be written as

$$\ell(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto -n \ \alpha \log \beta - n \log \Gamma(\alpha) - \frac{1}{\beta} \sum_{i=1}^{n} \frac{x_i}{1 - x_i} + \alpha \left[\sum_{i=1}^{n} \log(x_i) - \sum_{i=1}^{n} \log(1 - x_i) \right]. \tag{3}$$

From (3) we have the components of the score vector $\boldsymbol{U_\theta} = [U_\alpha, U_\beta]$ is given by

$$U_\alpha = \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log(1 - x_i) - n \log \beta - n \ \psi(\alpha) \tag{4}$$

$$U_\beta = \frac{1}{\beta^2} \sum_{i=1}^{n} \frac{x_i}{1 - x_i} - \frac{n \ \alpha}{\beta} \tag{5}$$

where $\psi(\cdot)$ denotes the digamma function, defined as $\psi(u) = \frac{d}{d \ u} \ \log \Gamma(u)$.

Setting $U_\alpha = 0$ and $U_\beta = 0$ and solving simultaneously for α and β we obtain the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively.

The expected Fisher information matrix of $\boldsymbol{\theta}$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta} \mid \boldsymbol{x}) = n \begin{bmatrix} \psi'(\alpha) & \frac{1}{\beta} \\ \frac{1}{\beta} & \frac{\alpha}{\beta^2} \end{bmatrix} \tag{6}$$

and the corresponding inverse matrix is

$$I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{x}) = \frac{1}{n} \begin{bmatrix} \dfrac{\alpha}{\psi'(\alpha)\ \alpha - 1} & -\dfrac{\beta}{\psi'(\alpha)\ \alpha - 1} \\ -\dfrac{\beta}{\psi'(\alpha)\ \alpha - 1} & \dfrac{\psi'(\alpha)\ \beta^2}{\psi'(\alpha)\ \alpha - 1} \end{bmatrix} \tag{7}$$

where $\psi'(\cdot)$ denotes the trigamma function, defined as $\psi'(u) = \frac{d}{du}\psi(u)$.

Existence and uniqueness of MLEs: The likelihood equations can be written from (4) and (5) as

$$\frac{d\ell}{d\alpha} = -n \log\beta - n\ \psi(\alpha) - \sum_{i=1}^{n}\ \log\left(\frac{x_i}{1-x_i}\right) = 0 \tag{8}$$

$$\text{and} \quad \frac{d\ell}{d\beta} = \frac{1}{\beta^2} \sum_{i=1}^{n} \frac{x_i}{1-x_i} - \frac{n\ \alpha}{\beta} = 0 \tag{9}$$

Now writing $\frac{x_i}{1-x_i} = y_i$ we can see that Eq. (9) yields $\hat{\beta} = \frac{\bar{y}}{\alpha}$. Substituting this in the Eq. (8) gives us

$$-\log\bar{y} + \log\alpha - \psi(\alpha) - \overline{\log y} = 0.$$

For given y, $-\log\bar{y} - \overline{\log y}$ is defined and fixed. Note that $\log\bar{y} \geq \overline{\log y}$, with the equality holding only when $y_1 = y_2 = \ldots = y_n$. But $Pr(y_1 = y_2 = \ldots = y_n) = 0$ thus we can take $\log\bar{y} > \overline{\log y}$. Now, $f(\alpha) = \log\alpha - \psi(\alpha)$ is continuous with $\lim_{\alpha\to\infty} f(\alpha) = 0$ and $\lim_{\alpha\to 0} f(\alpha) = \infty$. Hence the inverse of $f(\alpha)$ is non zero since $\alpha > 0$. Moreover $\frac{d}{d\alpha} f(\alpha) < 0$, thus there is a unique $\hat{\alpha}$ such that $\log\hat{\alpha} - \psi(\hat{\alpha}) = \log\bar{y} + \overline{\log y}$.

Hence the MLE exits and is unique.

Alternatively, one can show that the likelihood in Eq. (3) with β substituted by its estimate $\hat{\beta} = \frac{\bar{y}}{\alpha}$ given by

$$\begin{aligned} \ell(\alpha, \hat{\beta} \mid \boldsymbol{x}) &= -n\ \alpha \log\hat{\beta} + \alpha \sum_{i=1}^{n}\ \log\left(\frac{x_i}{1-x_i}\right) - \frac{1}{\hat{\beta}} \sum_{i=1}^{n} \frac{x_i}{1-x_i} - n\log\Gamma(\alpha) \\ &= n\ \alpha \log\alpha\bar{y} + \alpha \sum_{i=1}^{n}\ \log y_i - n\alpha - n\log\Gamma(\alpha) \end{aligned}$$

is strictly concave and hence will have unique maximum w.r.t. α.

3. Bias-corrected MLEs

While the MLE is a universally accepted method of parameter estimation having desirable large sample properties, its finite sample behavior especially the unwanted bias often rises questions. As such it is of interest for many practitioners to take recourse to the bias correction approaches to compute nearly unbiased estimates. In what follows we discuss three approaches for bias-correction of the maximum likelihood estimators of the parameters that index the GU distribution. At first, we shall consider the general formula introduced by Cox and Snell (1968). As noted by the authors, when the sample data are independent, but not necessarily identically distributed, the bias of the rth element of the MLE of $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}$, is calculated as

$$\mathcal{B}(\hat{\boldsymbol{\theta}}_r) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \boldsymbol{I}^{ri}\ \boldsymbol{I}^{jl}\ [0.5\boldsymbol{I}_{ijl} + \boldsymbol{I}_{ij,l}] + \mathcal{O}(n^{-2}), \tag{10}$$

where $r = 1, ..., p, \boldsymbol{I}^{ij}$ denotes the (i, j) th element of the inverse of the expected Fisher information matrix, $\boldsymbol{I}_{ijl} = \mathbb{E}\left[\frac{\partial^3 \ell}{\partial \theta_i\ \partial \theta_j\ \partial \theta_l}\right]$ and $\boldsymbol{I}_{ij,l} = \mathbb{E}\left[\left(\frac{\partial^2 \ell}{\partial \theta_i\ \partial \theta_j}\right)\ \left(\frac{\partial \ell}{\partial \theta_l}\right)\right]$.

For the GU distribution, after some algebra, we obtain $\boldsymbol{I}_{111} = -n\ \psi''(\alpha)$, $\boldsymbol{I}_{122} = \boldsymbol{I}_{221} = \boldsymbol{I}_{212} = \frac{n}{\beta^2}, \boldsymbol{I}_{222} = \frac{4\ n\ \alpha}{\beta^3}, \boldsymbol{I}_{22,1} = -\frac{2\ n}{\beta^2}, \boldsymbol{I}_{22,2} = -\frac{2\ n\ \alpha}{\beta^3}$ and all other terms are equal to zero, where $\psi''(\cdot)$ denotes the tetragamma function, defined as $\psi''(u) = \frac{d}{du}\psi'(u)$.

By replacing these terms in Eq. (10) we obtain the following expressions for the second order biases of $\hat{\alpha}$ and $\hat{\beta}$

$$\mathcal{B}(\hat{\alpha}) = \frac{0.5\ \psi'(\alpha)\ \alpha - 0.5\ \psi''(\alpha)\ \alpha^2 - 1}{n\left[\psi'(\alpha)\ \alpha - 1\right]^2} + \mathcal{O}(n^{-2}) \tag{11}$$

and

$$\mathcal{B}(\hat{\beta}) = \frac{\beta\left[\alpha\ \psi'(\alpha)^2 - 1.5\ \psi'(\alpha) - 0.5\ \psi''(\alpha), \alpha\right]}{n\left[\psi'(\alpha)\ \alpha - 1\right]^2} + \mathcal{O}(n^{-2}). \tag{12}$$

Thus, from (11) and (12) we may define the bias-corrected estimators (BCE) of $\hat{\alpha}$ and $\hat{\beta}$, $\hat{\alpha}_{BCE} = \hat{\alpha} - \hat{\mathcal{B}}(\hat{\alpha})$ and $\hat{\beta}_{BCE} = \hat{\beta} - \hat{\mathcal{B}}(\hat{\beta})$, respectively. It is expected that both $\hat{\alpha}_{BCE}$ and $\hat{\beta}_{BCE}$ will exhibit better properties than the uncorrected estimators $\hat{\alpha}$ and $\hat{\beta}$.

Additionally, as an alternative approach to the analytically bias corrected estimator, we may envisage the parametric Bootstrap methodology for bias reduction (PBE), which was introduced by Efron (1982). Here, for an arbitrary parameter $\boldsymbol{\theta}$, the estimated bias of $\hat{\boldsymbol{\theta}}$ is given by

$$\hat{\mathcal{B}}(\hat{\boldsymbol{\theta}}) = \frac{1}{B}\sum_{j=1}^{B}\ \hat{\boldsymbol{\theta}}_{(j)} - \hat{\boldsymbol{\theta}} \tag{13}$$

where $\hat{\boldsymbol{\theta}}_{(j)}$ is the MLE of θ obtained from the jth Bootstrap sample, generated from (1) and using the maximum likelihood estimate $\hat{\boldsymbol{\theta}}$ as the true value. Hence, the Bootstrap bias-corrected estimator is defined as

$$\hat{\boldsymbol{\theta}}_{PBE} = 2\hat{\boldsymbol{\theta}} - \frac{1}{B}\sum_{j=1}^{B}\ \hat{\boldsymbol{\theta}}_{(j)}. \tag{14}$$

For readers interested in more details about bias-corrected maximum likelihood estimators we recommend Cordeiro and Cribari-Neto (2014). Moreover, the mle.tools library (Mazucheli, Menezes, and Nadarajah 2017) in R software provide an efficient way to get bias corrected estimates by the methodology proposed by Cox and Snell (1968) for any probability density function.

An alternative approach to obtain bias-corrected estimators proposed by Firth (1993) consists of transforming the score vector $\boldsymbol{U}(\boldsymbol{\theta}) = \frac{\partial \ell}{\partial \boldsymbol{\theta}}$ before obtaining the maximum likelihood estimators. This method is known as the preventive method. The modified score vector is defined by

$$\boldsymbol{U}^*(\boldsymbol{\theta}) = \boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{I}(\boldsymbol{\theta})\ \mathcal{B}(\hat{\boldsymbol{\theta}}) \tag{15}$$

where $\boldsymbol{I}(\boldsymbol{\theta})$ is the expected information matrix and $\mathcal{B}(\hat{\boldsymbol{\theta}})$ is the second-order bias vector with the components defined in (11) and (12).

It is important to point out that when the second-order derivatives of the log-likelihood with respect to the parameters do not depend on the sample values, the bias-corrected estimators can be obtained by maximizing the modified log-likelihood $\ell^* = \ell + 0.5\ \log|\mathbf{I}(\boldsymbol{\theta})|$.

Table 1. Estimated bias (root mean-squared error) for α and β, ($\alpha = 0.5$).

		Estimator of α				Estimator of β			
β	n	MLE	BCE	PBE	FBC	MLE	BCE	PBE	FBC
0.3	10	0.151 (0.371)	−0.000 (0.243)	−0.061 (0.217)	−0.042 (0.192)	−0.023 (0.173)	−0.001 (0.184)	0.000 (0.186)	0.056 (0.193)
	20	0.060 (0.178)	−0.002 (0.145)	−0.013 (0.141)	−0.014 (0.139)	−0.010 (0.122)	0.001 (0.126)	0.002 (0.126)	0.024 (0.123)
	30	0.040 (0.132)	0.000 (0.115)	−0.004 (0.114)	−0.006 (0.113)	−0.007 (0.101)	0.000 (0.103)	0.001 (0.103)	0.014 (0.099)
	40	0.029 (0.107)	−0.000 (0.097)	−0.003 (0.096)	−0.004 (0.096)	−0.006 (0.086)	0.000 (0.088)	0.000 (0.088)	0.008 (0.085)
	50	0.023 (0.094)	0.000 (0.086)	−0.001 (0.086)	−0.002 (0.085)	−0.005 (0.078)	−0.001 (0.078)	−0.000 (0.078)	0.005 (0.076)
0.5	10	0.153 (0.372)	0.001 (0.244)	−0.060 (0.215)	−0.039 (0.198)	−0.040 (0.291)	−0.002 (0.309)	−0.001 (0.310)	0.104 (0.321)
	20	0.065 (0.181)	0.002 (0.146)	−0.009 (0.142)	−0.011 (0.143)	−0.019 (0.207)	0.000 (0.213)	0.001 (0.214)	0.042 (0.205)
	30	0.041 (0.132)	0.001 (0.114)	−0.003 (0.113)	−0.005 (0.113)	−0.012 (0.169)	0.001 (0.173)	0.001 (0.173)	0.024 (0.165)
	40	0.030 (0.108)	0.001 (0.097)	−0.002 (0.097)	−0.004 (0.095)	−0.008 (0.146)	0.002 (0.148)	0.002 (0.148)	0.016 (0.142)
	50	0.023 (0.093)	0.000 (0.086)	−0.001 (0.085)	−0.003 (0.084)	−0.006 (0.130)	0.001 (0.131)	0.002 (0.131)	0.011 (0.127)
1.5	10	0.147 (0.361)	−0.003 (0.237)	−0.064 (0.208)	−0.041 (0.206)	−0.118 (0.866)	−0.007 (0.921)	−0.004 (0.922)	0.324 (0.938)
	20	0.061 (0.177)	−0.002 (0.144)	−0.012 (0.140)	−0.015 (0.140)	−0.059 (0.613)	−0.001 (0.633)	−0.000 (0.634)	0.142 (0.606)
	30	0.039 (0.131)	−0.001 (0.114)	−0.005 (0.113)	−0.007 (0.113)	−0.041 (0.502)	−0.002 (0.513)	−0.001 (0.513)	0.078 (0.489)
	40	0.029 (0.107)	0.000 (0.097)	−0.002 (0.096)	−0.003 (0.097)	−0.036 (0.435)	−0.006 (0.441)	−0.006 (0.442)	0.052 (0.425)
	50	0.023 (0.093)	−0.000 (0.086)	−0.001 (0.086)	−0.002 (0.086)	−0.030 (0.390)	−0.007 (0.395)	−0.006 (0.395)	0.036 (0.383)
2.0	10	0.150 (0.381)	−0.001 (0.251)	−0.062 (0.218)	−0.043 (0.209)	−0.171 (1.143)	−0.023 (1.214)	−0.020 (1.213)	0.445 (1.264)
	20	0.062 (0.179)	−0.001 (0.145)	−0.011 (0.141)	−0.016 (0.138)	−0.087 (0.808)	−0.009 (0.833)	−0.008 (0.833)	0.197 (0.808)
	30	0.040 (0.132)	0.000 (0.114)	−0.004 (0.113)	−0.008 (0.111)	−0.065 (0.659)	−0.013 (0.673)	−0.012 (0.673)	0.105 (0.634)
	40	0.029 (0.107)	0.000 (0.096)	−0.002 (0.096)	−0.004 (0.095)	−0.047 (0.573)	−0.008 (0.582)	−0.007 (0.582)	0.069 (0.549)
	50	0.023 (0.093)	0.000 (0.086)	−0.001 (0.086)	−0.003 (0.085)	−0.035 (0.515)	−0.004 (0.521)	−0.003 (0.521)	0.041 (0.496)
3.0	10	0.147 (0.348)	−0.003 (0.227)	−0.064 (0.201)	−0.037 (0.214)	−0.251 (1.736)	−0.029 (1.844)	−0.028 (1.837)	0.686 (1.915)
	20	0.061 (0.179)	−0.002 (0.145)	−0.012 (0.141)	−0.012 (0.142)	−0.115 (1.230)	0.002 (1.270)	0.003 (1.270)	0.310 (1.232)
	30	0.039 (0.132)	−0.000 (0.114)	−0.005 (0.113)	−0.005 (0.113)	−0.085 (0.997)	−0.007 (1.017)	−0.006 (1.018)	0.175 (0.982)
	40	0.029 (0.108)	−0.001 (0.097)	−0.003 (0.096)	−0.002 (0.095)	−0.055 (0.875)	0.004 (0.889)	0.005 (0.890)	0.106 (0.847)
	50	0.022 (0.093)	−0.001 (0.086)	−0.002 (0.085)	−0.000 (0.085)	−0.041 (0.781)	0.007 (0.791)	0.008 (0.792)	0.065 (0.749)

Table 2. Estimated bias (root mean-squared error) for α and β, ($\alpha = 1.0$).

		Estimator of α				Estimator of β			
β	n	MLE	BCE	PBE	FBC	MLE	BCE	PBE	FBC
0.3	10	0.343 (0.844)	−0.005 (0.544)	−0.148 (0.492)	−0.111 (0.339)	−0.026 (0.150)	−0.001 (0.160)	−0.001 (0.160)	0.033 (0.162)
	20	0.141 (0.398)	−0.003 (0.318)	−0.028 (0.310)	−0.025 (0.274)	−0.013 (0.106)	−0.000 (0.110)	−0.000 (0.110)	0.008 (0.109)
	30	0.086 (0.284)	−0.005 (0.245)	−0.015 (0.243)	−0.004 (0.232)	−0.008 (0.087)	0.001 (0.089)	0.001 (0.089)	0.001 (0.088)
	40	0.061 (0.231)	−0.005 (0.207)	−0.011 (0.206)	−0.001 (0.204)	−0.006 (0.076)	0.001 (0.077)	0.001 (0.077)	0.000 (0.076)
	50	0.049 (0.203)	−0.003 (0.185)	−0.007 (0.185)	0.000 (0.181)	−0.005 (0.068)	0.001 (0.069)	0.001 (0.069)	0.000 (0.068)
0.5	10	0.354 (0.851)	0.003 (0.546)	−0.143 (0.488)	−0.105 (0.357)	−0.049 (0.245)	−0.008 (0.260)	−0.007 (0.261)	0.058 (0.273)
	20	0.147 (0.404)	0.003 (0.322)	−0.023 (0.313)	−0.029 (0.281)	−0.024 (0.176)	−0.003 (0.182)	−0.002 (0.182)	0.018 (0.182)
	30	0.092 (0.286)	0.001 (0.244)	−0.009 (0.242)	−0.011 (0.237)	−0.015 (0.144)	−0.001 (0.147)	−0.001 (0.147)	0.008 (0.147)
	40	0.067 (0.232)	0.001 (0.206)	−0.004 (0.205)	−0.006 (0.204)	−0.013 (0.125)	−0.001 (0.127)	−0.001 (0.127)	0.005 (0.128)
	50	0.054 (0.202)	0.002 (0.183)	−0.002 (0.182)	−0.005 (0.183)	−0.011 (0.112)	−0.002 (0.113)	−0.002 (0.113)	0.004 (0.115)
1.5	10	0.343 (0.813)	−0.005 (0.520)	−0.152 (0.455)	−0.111 (0.372)	−0.135 (0.748)	−0.012 (0.796)	−0.010 (0.798)	0.205 (0.813)
	20	0.145 (0.395)	0.001 (0.314)	−0.024 (0.306)	−0.037 (0.283)	−0.074 (0.530)	−0.009 (0.547)	−0.009 (0.548)	0.084 (0.539)
	30	0.090 (0.284)	−0.001 (0.244)	−0.011 (0.241)	−0.018 (0.235)	−0.049 (0.436)	−0.005 (0.445)	−0.005 (0.445)	0.048 (0.435)
	40	0.068 (0.236)	0.001 (0.210)	−0.004 (0.209)	−0.009 (0.204)	−0.039 (0.379)	−0.006 (0.385)	−0.006 (0.385)	0.028 (0.377)
	50	0.054 (0.204)	0.001 (0.186)	−0.002 (0.185)	−0.005 (0.184)	−0.031 (0.339)	−0.004 (0.344)	−0.004 (0.344)	0.018 (0.337)
2.0	10	0.346 (0.842)	−0.003 (0.541)	−0.151 (0.472)	−0.120 (0.381)	−0.179 (1.000)	−0.014 (1.066)	−0.011 (1.067)	0.288 (1.079)
	20	0.142 (0.391)	−0.002 (0.311)	−0.028 (0.304)	−0.040 (0.286)	−0.089 (0.712)	−0.003 (0.736)	−0.002 (0.737)	0.116 (0.705)
	30	0.089 (0.286)	−0.002 (0.246)	−0.012 (0.243)	−0.017 (0.233)	−0.054 (0.586)	0.005 (0.600)	0.005 (0.601)	0.057 (0.572)
	40	0.064 (0.234)	−0.002 (0.209)	−0.007 (0.208)	−0.008 (0.202)	−0.039 (0.506)	0.005 (0.515)	0.006 (0.515)	0.037 (0.500)
	50	0.051 (0.205)	−0.001 (0.187)	−0.004 (0.186)	−0.004 (0.180)	−0.032 (0.453)	0.004 (0.460)	0.004 (0.460)	0.019 (0.447)
3.0	10	0.350 (0.837)	−0.000 (0.537)	−0.149 (0.468)	−0.123 (0.386)	−0.275 (1.510)	−0.028 (1.608)	−0.024 (1.612)	0.521 (1.694)
	20	0.141 (0.396)	−0.003 (0.316)	−0.028 (0.308)	−0.043 (0.283)	−0.130 (1.070)	0.000 (1.106)	0.001 (1.107)	0.212 (1.085)
	30	0.085 (0.283)	−0.006 (0.244)	−0.016 (0.242)	−0.022 (0.230)	−0.075 (0.880)	0.014 (0.901)	0.014 (0.901)	0.116 (0.867)
	40	0.062 (0.233)	−0.004 (0.208)	−0.009 (0.207)	−0.011 (0.202)	−0.057 (0.757)	0.010 (0.771)	0.010 (0.771)	0.073 (0.755)
	50	0.050 (0.202)	−0.002 (0.184)	−0.005 (0.184)	−0.006 (0.182)	−0.046 (0.684)	0.008 (0.694)	0.008 (0.694)	0.046 (0.676)

Table 3. Estimated bias (root mean-squared error) for α and β, ($\alpha = 2.0$).

		Estimator of α				Estimator of β			
β	n	MLE	BCE	PBE	FBC	MLE	BCE	PBE	FBC
0.3	10	0.782 (1.839)	0.009 (1.167)	−0.303 (1.072)	−0.361 (0.677)	−0.029 (0.139)	−0.003 (0.148)	−0.003 (0.149)	0.057 (0.158)
	20	0.322 (0.874)	0.004 (0.691)	−0.052 (0.672)	−0.136 (0.532)	−0.015 (0.099)	−0.001 (0.103)	−0.001 (0.103)	0.023 (0.102)
	30	0.207 (0.629)	0.006 (0.535)	−0.016 (0.529)	−0.064 (0.463)	−0.010 (0.082)	−0.001 (0.084)	−0.001 (0.084)	0.011 (0.082)
	40	0.146 (0.509)	0.000 (0.451)	−0.012 (0.448)	−0.038 (0.414)	−0.007 (0.071)	0.000 (0.073)	0.000 (0.073)	0.007 (0.071)
	50	0.116 (0.441)	0.002 (0.400)	−0.006 (0.399)	−0.027 (0.369)	−0.006 (0.064)	−0.000 (0.065)	−0.000 (0.065)	0.005 (0.063)
0.5	10	0.775 (1.780)	0.004 (1.123)	−0.317 (1.014)	−0.354 (0.678)	−0.051 (0.229)	−0.008 (0.243)	−0.008 (0.244)	0.091 (0.263)
	20	0.329 (0.859)	0.010 (0.675)	−0.046 (0.657)	−0.120 (0.542)	−0.027 (0.165)	−0.004 (0.171)	−0.004 (0.171)	0.031 (0.169)
	30	0.208 (0.614)	0.008 (0.521)	−0.015 (0.515)	−0.053 (0.467)	−0.018 (0.136)	−0.002 (0.139)	−0.002 (0.139)	0.014 (0.137)
	40	0.150 (0.503)	0.004 (0.445)	−0.008 (0.442)	−0.029 (0.412)	−0.012 (0.118)	−0.001 (0.120)	−0.001 (0.120)	0.008 (0.118)
	50	0.118 (0.432)	0.003 (0.391)	−0.004 (0.390)	−0.017 (0.372)	−0.010 (0.105)	−0.001 (0.106)	−0.001 (0.106)	0.005 (0.106)
1.5	10	0.771 (1.810)	0.001 (1.148)	−0.326 (1.007)	−0.278 (0.697)	−0.147 (0.685)	−0.016 (0.731)	−0.016 (0.732)	0.203 (0.774)
	20	0.323 (0.864)	0.005 (0.682)	−0.051 (0.663)	−0.070 (0.571)	−0.077 (0.488)	−0.008 (0.505)	−0.008 (0.505)	0.056 (0.512)
	30	0.203 (0.618)	0.003 (0.526)	−0.019 (0.521)	−0.017 (0.492)	−0.051 (0.402)	−0.005 (0.411)	−0.004 (0.411)	0.017 (0.412)
	40	0.149 (0.502)	0.003 (0.444)	−0.009 (0.441)	−0.010 (0.431)	−0.040 (0.349)	−0.005 (0.355)	−0.005 (0.355)	0.013 (0.359)
	50	0.114 (0.428)	−0.000 (0.388)	−0.008 (0.387)	−0.002 (0.389)	−0.030 (0.312)	−0.002 (0.316)	−0.001 (0.316)	0.005 (0.321)
2.0	10	0.779 (1.819)	0.007 (1.152)	−0.323 (1.006)	−0.261 (0.710)	−0.192 (0.932)	−0.018 (0.997)	−0.017 (0.999)	0.260 (1.039)
	20	0.321 (0.862)	0.004 (0.681)	−0.052 (0.663)	−0.072 (0.571)	−0.095 (0.662)	−0.003 (0.685)	−0.003 (0.685)	0.075 (0.682)
	30	0.208 (0.620)	0.008 (0.526)	−0.015 (0.520)	−0.031 (0.491)	−0.068 (0.545)	−0.006 (0.558)	−0.006 (0.558)	0.038 (0.553)
	40	0.149 (0.498)	0.003 (0.439)	−0.009 (0.437)	−0.017 (0.424)	−0.049 (0.466)	−0.002 (0.474)	−0.001 (0.474)	0.022 (0.474)
	50	0.120 (0.429)	0.005 (0.388)	−0.003 (0.386)	−0.011 (0.384)	−0.042 (0.416)	−0.004 (0.421)	−0.004 (0.422)	0.016 (0.427)
3.0	10	0.803 (1.862)	0.024 (1.178)	−0.310 (1.019)	−0.262 (0.725)	−0.329 (1.382)	−0.071 (1.468)	−0.071 (1.469)	0.406 (1.544)
	20	0.326 (0.872)	0.007 (0.689)	−0.049 (0.670)	−0.071 (0.576)	−0.158 (1.000)	−0.020 (1.033)	−0.019 (1.035)	0.111 (1.007)
	30	0.199 (0.616)	−0.001 (0.526)	−0.023 (0.520)	−0.027 (0.484)	−0.102 (0.810)	−0.008 (0.829)	−0.008 (0.829)	0.044 (0.815)
	40	0.144 (0.497)	−0.002 (0.440)	−0.014 (0.438)	−0.013 (0.429)	−0.076 (0.703)	−0.005 (0.715)	−0.005 (0.716)	0.023 (0.708)
	50	0.108 (0.421)	−0.006 (0.383)	−0.014 (0.382)	−0.007 (0.387)	−0.055 (0.625)	0.002 (0.634)	0.002 (0.635)	0.015 (0.639)

Table 4. Estimated bias (root mean-squared error) for α and β, ($\alpha = 3.0$).

		Estimator of α				Estimator of β			
β	n	MLE	BCE	PBE	FBC	MLE	BCE	PBE	FBC
0.3	10	1.165 (2.707)	−0.021 (1.712)	−0.493 (1.626)	−0.701 (0.983)	−0.028 (0.135)	−0.002 (0.145)	−0.002 (0.145)	0.070 (0.160)
	20	0.494 (1.337)	0.002 (1.056)	−0.085 (1.030)	−0.295 (0.729)	−0.014 (0.098)	0.000 (0.102)	0.000 (0.102)	0.027 (0.098)
	30	0.310 (0.958)	0.000 (0.816)	−0.035 (0.808)	−0.150 (0.641)	−0.009 (0.081)	0.000 (0.083)	0.000 (0.083)	0.014 (0.078)
	40	0.229 (0.775)	0.003 (0.685)	−0.016 (0.681)	−0.084 (0.588)	−0.007 (0.070)	−0.000 (0.071)	−0.000 (0.071)	0.008 (0.068)
	50	0.184 (0.665)	0.005 (0.601)	−0.006 (0.599)	−0.051 (0.545)	−0.006 (0.062)	−0.001 (0.063)	−0.001 (0.063)	0.005 (0.061)
0.5	10	1.245 (2.894)	0.035 (1.830)	−0.459 (1.696)	−0.647 (0.998)	−0.053 (0.228)	−0.009 (0.243)	−0.009 (0.244)	0.107 (0.263)
	20	0.502 (1.316)	0.008 (1.035)	−0.080 (1.007)	−0.270 (0.766)	−0.026 (0.161)	−0.002 (0.166)	−0.002 (0.166)	0.042 (0.165)
	30	0.317 (0.942)	0.007 (0.798)	−0.029 (0.789)	−0.128 (0.678)	−0.018 (0.132)	−0.002 (0.135)	−0.002 (0.135)	0.020 (0.131)
	40	0.232 (0.764)	0.005 (0.673)	−0.013 (0.669)	−0.070 (0.609)	−0.013 (0.114)	−0.001 (0.116)	−0.001 (0.117)	0.011 (0.113)
	50	0.178 (0.659)	0.000 (0.596)	−0.012 (0.594)	−0.038 (0.572)	−0.010 (0.103)	−0.000 (0.104)	−0.000 (0.104)	0.007 (0.103)
1.5	10	1.168 (2.739)	−0.019 (1.736)	−0.520 (1.544)	−0.615 (1.016)	−0.139 (0.676)	−0.006 (0.725)	−0.006 (0.725)	0.311 (0.786)
	20	0.485 (1.299)	−0.006 (1.025)	−0.093 (0.999)	−0.212 (0.787)	−0.069 (0.484)	0.002 (0.502)	0.002 (0.503)	0.100 (0.489)
	30	0.306 (0.941)	−0.004 (0.801)	−0.039 (0.792)	−0.097 (0.686)	−0.047 (0.394)	0.001 (0.404)	0.001 (0.405)	0.047 (0.396)
	40	0.224 (0.764)	−0.002 (0.676)	−0.021 (0.672)	−0.044 (0.623)	−0.036 (0.342)	0.000 (0.349)	0.001 (0.349)	0.023 (0.343)
	50	0.181 (0.663)	0.003 (0.599)	−0.009 (0.598)	−0.019 (0.572)	−0.031 (0.306)	−0.002 (0.311)	−0.002 (0.311)	0.012 (0.310)
2.0	10	1.199 (2.793)	0.002 (1.767)	−0.506 (1.561)	−0.575 (1.026)	−0.196 (0.904)	−0.018 (0.967)	−0.018 (0.968)	0.383 (1.045)
	20	0.500 (1.335)	0.007 (1.053)	−0.081 (1.025)	−0.166 (0.827)	−0.101 (0.648)	−0.008 (0.670)	−0.008 (0.671)	0.103 (0.658)
	30	0.309 (0.942)	−0.001 (0.802)	−0.036 (0.793)	−0.070 (0.721)	−0.066 (0.530)	−0.003 (0.543)	−0.002 (0.543)	0.047 (0.537)
	40	0.221 (0.756)	−0.005 (0.669)	−0.023 (0.665)	−0.035 (0.635)	−0.049 (0.460)	−0.001 (0.468)	−0.001 (0.469)	0.024 (0.462)
	50	0.182 (0.659)	0.003 (0.596)	−0.008 (0.594)	−0.019 (0.578)	−0.043 (0.413)	−0.004 (0.418)	−0.004 (0.418)	0.013 (0.413)
3.0	10	1.185 (2.765)	−0.007 (1.750)	−0.517 (1.535)	−0.478 (1.087)	−0.263 (1.387)	0.006 (1.493)	0.008 (1.495)	0.515 (1.589)
	20	0.477 (1.329)	−0.013 (1.055)	−0.100 (1.029)	−0.121 (0.873)	−0.129 (0.979)	0.012 (1.018)	0.013 (1.019)	0.140 (1.019)
	30	0.299 (0.943)	−0.010 (0.806)	−0.045 (0.798)	−0.050 (0.742)	−0.087 (0.804)	0.009 (0.825)	0.009 (0.825)	0.067 (0.817)
	40	0.216 (0.767)	−0.010 (0.681)	−0.028 (0.678)	−0.021 (0.656)	−0.063 (0.698)	0.010 (0.712)	0.010 (0.712)	0.035 (0.706)
	50	0.169 (0.657)	−0.008 (0.597)	−0.020 (0.595)	−0.010 (0.593)	−0.052 (0.622)	0.007 (0.631)	0.007 (0.632)	0.022 (0.632)

Table 5. MLEs and bias-corrected MLEs (Bootstrap standard-error).

Estimators	α	β
MLE	1.3086 (0.1364)	0.2278 (0.0282)
BCE	1.2874 (0.1349)	0.2292 (0.0280)
PBE	1.2875 (0.1342)	0.2291 (0.0279)
FBC	1.2873 (0.1311)	0.2292 (0.0277)

Table 6. Statistics (Bootstrap *p*-values) associated with goodness-of-fit measures.

Estimators	KS	CvM	AD
MLE	0.0807 (0.2553)	0.2305 (0.2155)	1.1976 (0.2684)
BCE	0.0742 (0.3485)	0.2061 (0.2560)	1.1345 (0.2937)
PBE	0.0751 (0.3348)	0.2100 (0.2490)	1.1447 (0.2895)
FBC	0.0743 (0.3470)	0.2066 (0.2552)	1.1357 (0.2932)

4. Simulation study

In this section, we conducted a simulation study to evaluate the finite-sample behavior of the MLEs of α and β and their bias-corrected counterparts obtained by Cox-Snell methodology (BCE), Firth's Bias Correction (FBC) method and parametric Bootstrap approach (PBE). We have drawn random samples of size $n = 10, 20, 30, 40,$ and 50 and the values of the parameters were fixed at $\alpha = 0.5, 1.0, 2.0,$ and 3.0 and $\beta = 0.3, 0.5, 1.5, 2.0,$ and 3.0. The pseudo-random samples from Gamma-Uniform distribution were simulated using the transformation $X = Y/(1 + Y)$, where $Y \sim \text{Gamma}(\alpha, \beta)$. The number of Monte Carlo replications in each experiment was set at $M = 10,000$ and the number of Bootstrap replications was $B = 1,000$. All simulation were conducted in Ox Console (Doornik 2007) using the MaxBFGS function to obtain the maximum likelihood estimates of α and β. Tables 1–4 report the results of the simulations, that is, the estimated biases and root mean-squared errors for the investigated estimators.

From Table 2, we can see that the MLEs of β are highly biased, while for α, bias is moderate, especially when the sample size is small. For instance, when $n = 10$, $\alpha = 1.0$, the biases of the MLEs of α and β are approximately 0.3430 and -0.0258. On the other hand, in the same scenario, the biases of $\hat{\alpha}_{BCE}, \hat{\alpha}_{PBE}, \hat{\alpha}_{FBC}, \hat{\beta}_{BCE}, \hat{\beta}_{PBE}$, and $\hat{\beta}_{FBC}$ are approximately -0.0049, -0.1483, -0.1110, -0.0009, -0.0006, and 0.0330, respectively. We also observe similar results in Tables 1, 3, and 4. Hence, it is apparent from the simulations results that the estimators $\hat{\alpha}_{BCE}, \hat{\beta}_{BCE}, \hat{\alpha}_{PBE}, \hat{\beta}_{BCE}$ and $\hat{\alpha}_{FBC}, \hat{\beta}_{FBC}$ clearly outperform the corresponding MLEs. Indeed, the proposed estimators achieve substantial bias reduction, particularly for the small and moderate sample sizes and therefore may be considered as better alternatives of the MLEs for α and β.

In fact, a check reveals that the relative bias of MLEs are substantially higher compared to BCE for all sample sizes not just for $n = 10$. For example, with $n = 30$, from Tables 1 and 4 we can observe the following.

- when $\alpha = 0.5, \beta = 0.3,$ the relative bias for MLE(α) is 8%, BCE(α) is 0% and MLE(β) is 2.3%, BCE(β) is 0%.
- when $\alpha = 0.5, \beta = 3.0,$ the relative bias for MLE(α) is 7.8%, BCE(α) is 0% and MLE(β) is 2.8%, BCE(β) is 0.23%.
- when $\alpha = 3.0, \beta = 0.3,$ the relative bias for MLE(α) is 10%, BCE(α) is 0% and MLE(β) is 3%, BCE(β) is 0%.
- when $\alpha = 3.0, \beta = 3.0,$ the relative bias for MLE(α) is 10%, BCE(α) is 0.3% and MLE(β) is 3%, BCE(β) is 0.3%.

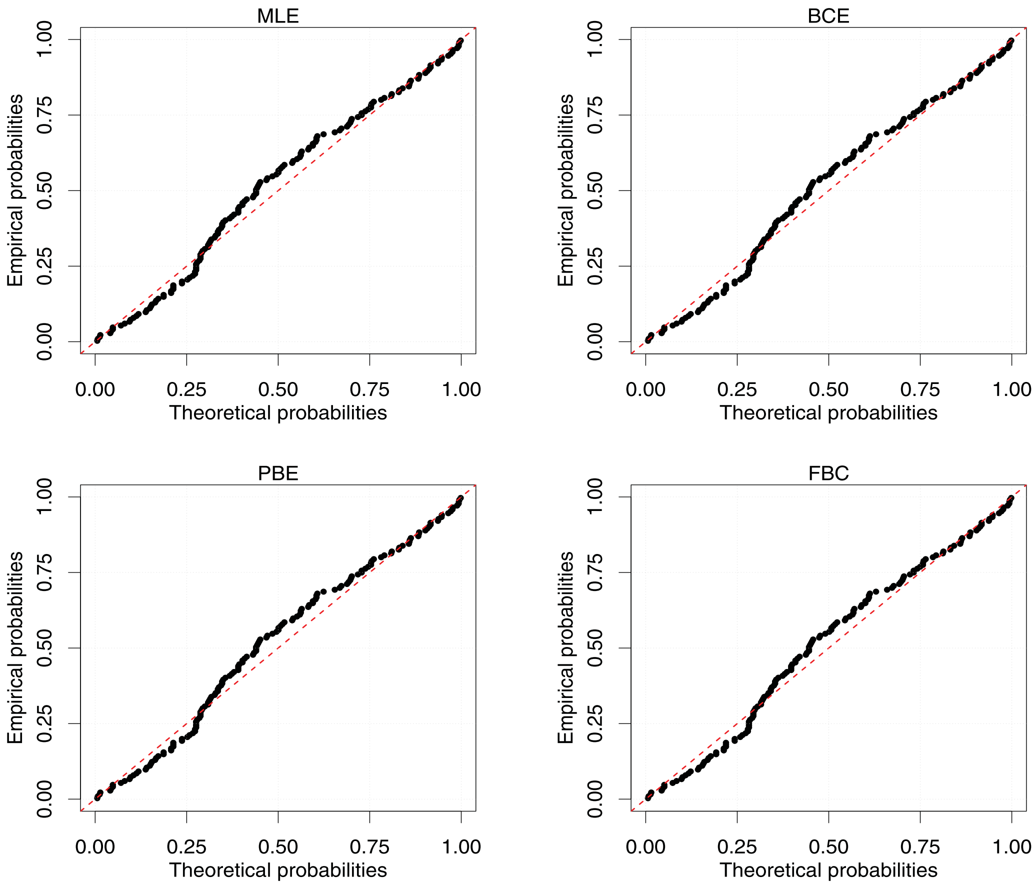

Figure 2. PP-Plots for the MLE, BCE, PBE, and FBC estimators.

Additionally, the root-mean squared errors of the corrected estimates are smaller than those of the uncorrected estimates. As expected, we may note that the root mean-squared errors decrease as the sample size increase.

5. Empirical example

In this section, we illustrate the practical impact of using the bias corrected estimators of the parameters associated to the Gamma-Uniform distribution. The GU distribution is considered as an attempt to adequately model the proportion of the total labor force ages 15–24 unemployed for 158 countries in 2018. The data set is available at http://www.indexmundi.com. Since the data was measured in the unit interval, the Gamma-Uniform distribution proposed here seems be a good alternative to model this data set. The mean and variance are, respectively, $\bar{x} = 0.201$ total labor force and $s^2 = 0.018$ total labor force^2, which evidences under-dispersion.

In Table 5, we report the MLEs of α and β. The corresponding bias corrections estimates, namely, the BCE, PBE, and FBC are also reported, with Bootstrapped standard errors based on 20,000 bootstrap samples (Efron and Tibshirani 1993). We can see that the bias-corrected estimates have smaller standard errors for both parameters, which means that the BCE, PBE, and FBC provide more accurate estimates.

In order to test whether the Gamma-Uniform distribution is adequate to describe the data, we evaluated the goodness-of-fit based on Kolmogorov-Smirnov statistic (KS), Anderson-Darling statistic (AD), and Cramér–von Mises statistics (CvM). Here the p-values are obtained using the Bootstrap samples, since the parameters were estimated (see, e.g. Durbin 1987). The results are presented in Table 6.

From Table 6 it can be seen that the values of all the goodness of fit criteria are almost smaller when bias corrected estimates were used compared to when actual maximum likelihood estimate was used. Corresponding higher p-values of these statistics give indication of the attainment of higher observed level of significance with bias corrections. These conclusions further supported by the PP-Plots presented in Figure 2.

6. Conclusion

The Gamma-Uniform distribution has been recently introduced by Torabi and Hedesh (2012), however, it did not receive much attention in the statistical literature. In this article, we have derived explicit expressions for the second-order biases of the MLEs of the parameters associated to the GU distribution and also preventive bias correction of Firth (1993). In addition, we have also considered the an alternative bias-correction using the parametric Bootstrap. Our simulation results support the use of analytical bias-correction for the MLEs, once bias, efficiency and computational cost are taken into account. An empirical application shows that the bias-corrected estimates (BCE, PBE, and FBC) are preferred, in terms of accuracy and goodness-of-fit, over the uncorrected estimates.

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