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A New Quantile Regression for Modeling Bounded Data under a Unit Birnbaum–Saunders Distribution with Applications in Medicine and Politics

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Abstract: Quantile regression provides a framework for modeling the relationship between a response variable and covariates using the quantile function. This work proposes a regression model for continuous variables bounded to the unit interval based on the unit Birnbaum–Saunders distribution as an alternative to the existing quantile regression models. By parameterizing the unit Birnbaum–Saunders distribution in terms of its quantile function allows us to model the effect of covariates across the entire response distribution, rather than only at the mean. Our proposal, especially useful for modeling quantiles using covariates, in general outperforms the other competing models available in the literature. These findings are supported by Monte Carlo simulations and applications using two real data sets. An R package, including parameter estimation, model checking as well as density, cumulative distribution, quantile and random number generating functions of the unit Birnbaum–Saunders distribution was developed and can be readily used to assess the suitability of our proposal.

Keywords: Birnbaum–Saunders distributions; data science; Monte Carlo simulations; R software; statistical modeling

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1. Introduction

Recently, researchers are interested in deriving distributions with bounded support. Some of these distributions are the unit Birnbaum–Saunders (UBS) [1], unit gamma [2], unit Gompertz [3], unit inverse-Gaussian [4], unit Lindley [5,6], unit Weibull (UWEI) [7–9], and trapezoidal beta [10] models, among others.

This interest is due to several natural and anthropogenic phenomena are measured as indexes, percentages, proportions, rates and ratios, which are bounded on a certain interval, usually the unit interval. The need for modeling and analyzing bounded data occurs in many fields of real life such as in medicine [8], politics [11], and psychology [12]. Hence, to statistically describe this kind of data, distributions with bounded support are needed.

A series of distributions with support on the unit interval have been proposed in the literature based on transformations of the cumulative distribution function (CDF). Among them, we can mention the log-Bilai [13], log-shifted Gompertz [14], unit Gompertz [3], unit inverse-Gaussian [4], unit Lindley [5], and UWEI [8] distributions.

In order to assess the influence of one or more covariates on the mean of the distribution of a response variable bounded on the unit interval, the following distributions can be used: beta [15], beta rectangular [16], log-Bilai [13], log-Lindley [17], log-weighted exponential [18], quasi-beta [19], simplex [20], unit gamma [2], and unit Lindley [5,6,21]. To investigate the relationship between the response mode and covariates, we can consider the beta [22], Kumaraswamy (KUMA), unit gamma, and unit Gompertz [23] distributions.

The challenges in applied regression have been changing considerably, and full statistical modeling rather than predicting just means and modes is required in many applications [24]. Some parametric quantile regression models for bounded response that are available in the literature are based on the exponentiated arcsech-normal [25], Johnson-t [26], KUMA [27], log-extended exponential-geometric (LEEG) [28], L-logistic [29], power Johnson SB [30], unit Chen (UCHE) [31], unit half-normal (UHN) [32], unit Burr-XII (UBUR) [33], and UWEI [8] distributions. Unlike regression models through the mean, quantile regression, introduced in [34], allows one to model the effect of covariates across the entire response distribution. This motivates us to propose new quantile regression models.

Originated from problems of vibration in commercial aircraft that caused fatigue in the materials, Birnbaum and Saunders [35] introduced a skew positive distribution that has been extensively studied in the statistical literature. During the last decade, the Birnbaum–Saunders (BS) distribution has received significant attention by many researchers who generalized the BS distribution [36], studied its mathematical and statistical properties [37,38], estimated its parameters [39], and conducted regression modeling [40,41] and its diagnostic analysis [42]. BS quantile regression models and its diagnostics have been recently studied in [42–45].

As mentioned, an extension of the BS distribution for modeling bounded data was proposed in [1]. The authors showed that the UBS distribution is flexible and could be a good alternative to the beta and KUMA distributions for modeling data supported on the unit interval. In this context, considering the lack of agreement on preference and advantage of a specific model to describe bounded phenomena in a full statistical fashion, based on [42], we parameterize the UBS distribution using its quantile function (QF) to introduce a parametric quantile regression model. It is well-known that there are at least three approaches to modeling quantiles conditional on covariates, which are: (i) the distribution-free approach [34]; (ii) the approach based on a pseudo-likelihood through an asymmetric Laplace distribution [46]; and (iii) the parametric approach with the traditional maximum likelihood (ML) framework. The current manuscript is classified as the third category. To the best of our knowledge, quantile regression for modeling bounded data under a UBS distribution have not been proposed until now.

In this paper, we parameterize the UBS distribution in terms of its QF to evaluate the influence of one or more covariates on any quantile of the distribution of the response variable. This strategy was considered in [8,25–29,47] also to model responses on the standard unit interval. Other strategies were considered in [30,48,49]. For discrete and continuous positive responses, the literature is scarce and we can cite [42,50,51], who considered the discrete generalized half-normal distribution for discrete responses. Therefore, the objective of this investigation is to propose, derive and apply a UBS quantile regression model as an alternative to the existing quantile regression models.

The remainder of the paper is as follows. In Section 2, we introduce and characterize the UBS distribution in terms of its QF. The estimation method of parameters for the UBS distribution, based on the ML method, is discussed also here. The UBS quantile regression model is formulated in Section 3. A simulation study is carried out to evaluate the performance of the proposed results which are presented in Section 4. Applications considering two real data sets related to political sciences and sports medicine with seven competing models to the UBS quantile regression are analyzed in Sections 5 and 6. Finally, Section 7 provides some conclusions and limitations of this work, as well as ideas for future investigation.

2. The Unit Birnbaum–Saunders Distribution

In this section, we parameterize the BS distribution in terms of its QF.

Considering the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{BS}(\alpha, \beta)$, the UBS distribution was proposed in [1] with probability density function (PDF) and CDF written, respectively, as

$$f(x;\alpha,\beta)=\frac{1}{2x\alpha\beta\sqrt{2\pi}}\left[\left(-\frac{\beta}{\log(x)}\right)^{\frac{1}{2}}+\left(-\frac{\beta}{\log(x)}\right)^{\frac{3}{2}}\right]\exp\left[\frac{1}{2\alpha^2}\left(2+\frac{\log(x)}{\beta}+\frac{\beta}{\log(x)}\right)\right] \quad (1)$$

and

$$F(x;\alpha,\beta)=1-\Phi\left\{\frac{1}{\alpha}\left[\left(-\frac{\log(x)}{\beta}\right)^{\frac{1}{2}}-\left(-\frac{\beta}{\log(x)}\right)^{\frac{1}{2}}\right]\right\}, \quad (2)$$

where $0<x<1$, $\alpha>0$ is the shape parameter and Φ is the CDF of the standard normal distribution. It is noteworthy that $\delta=\exp(-\beta)$ is a scale parameter and is also the median of the distribution of X, since $F(\delta;\alpha,\beta)=0.5$. In addition, the r-th moment of X is given by

$$\begin{aligned}\mathrm{E}(X^r) &= \int_0^1 x^r f(x;\alpha,\beta)\mathrm{d}x\\ &= f(x;\alpha,\beta)\tfrac{x^{r+1}}{r+1}\Big|_1^0-\int_0^1\frac{x^{r+1}}{r+1}f'(x;\alpha,\beta)\mathrm{d}x\\ &= \frac{1}{2(2r\alpha^2\beta+1)}\left[2r\alpha^2\beta+\left(2r\alpha^2\beta+1\right)^{\frac{1}{2}}+1\right]\exp\left[-\frac{(2r\alpha^2\beta+1)^{\frac{1}{2}}-1}{\alpha^2}\right],\end{aligned}$$

where $f'(x;\alpha,\beta)=\mathrm{d}f(x;\alpha,\beta)/\mathrm{d}x$.

By inverting the CDF defined in (2), we have the QF stated as

$$Q(\tau;\alpha,\beta)=\exp\left\{-\frac{2\beta}{2+[\alpha\Phi^{-1}(1-\tau)]^2-\alpha\Phi^{-1}(1-\tau)\sqrt{4+[\alpha\Phi^{-1}(1-\tau)]^2}}\right\}, \quad (3)$$

where $0<\tau<1$ and Φ^{-1} is the QF of the standard normal distribution. Now, by solving the linear equation $Q(\tau;\alpha,\beta)-\mu=0$ in β obtained from (3), we have

$$\beta=g^{-1}(\mu)=\log(\mu)h(\alpha,\tau), \quad (4)$$

where $h(\alpha,\tau)=-(1/2)\{2+[\alpha\Phi^{-1}(1-\tau)]^2-\alpha\Phi^{-1}(1-\tau)\sqrt{4+\alpha\Phi^{-1}(1-\tau)}\}$, and g^{-1} is the inverse function of g, which is assumed to be a quantile link function being strictly increasing, twice differentiable and mapping $(0,1)$ into $\mathbb{R}$.

Hence, we can parameterize (1) and (2), as a function of μ, by means of

$$\begin{aligned}f(x;\mu,\alpha,\tau) &= \frac{1}{2x\alpha\log(\mu)h(\alpha,\tau)\sqrt{2\pi}}\left\{\left[-\frac{\log(\mu)h(\alpha,\tau)}{\log(x)}\right]^{\frac{1}{2}}+\left[-\frac{\log(\mu)h(\alpha,\tau)}{\log(x)}\right]^{\frac{3}{2}}\right\}\\ &\quad\times\exp\left\{\frac{1}{2\alpha^2}\left[\frac{\log(x)}{\log(\mu)h(\alpha,\tau)}+\frac{\log(\mu)h(\alpha,\tau)}{\log(x)}+2\right]\right\} \quad (5)\end{aligned}$$

and

$$F(x;\mu,\alpha,\tau)=1-\Phi\left\{\frac{1}{\alpha}\left[\left(-\frac{\log(x)}{\log(\mu)h(\alpha,\tau)}\right)^{\frac{1}{2}}-\left(-\frac{\log(\mu)h(\alpha,\tau)}{\log(x)}\right)^{\frac{1}{2}}\right]\right\}, \quad (6)$$

where $0<x<1$, $0<\mu<1$, $\alpha>0$, and τ is fixed.

Figure 1 shows some possible shapes of the PDF of the UBS distribution for selected values of the parameters μ, α and τ. The behaviors of the mean, variance, coefficient of skewness and coefficient of kurtosis are shown in Figure 2.

An advantage of the parameterization stated in (5) and (6) is that μ is the τ-th quantile and, consequently, the interpretation of this parameterization becomes more interesting in applications. Furthermore, it is possible to consider that the parameter μ depends on covariates; see Section 3.

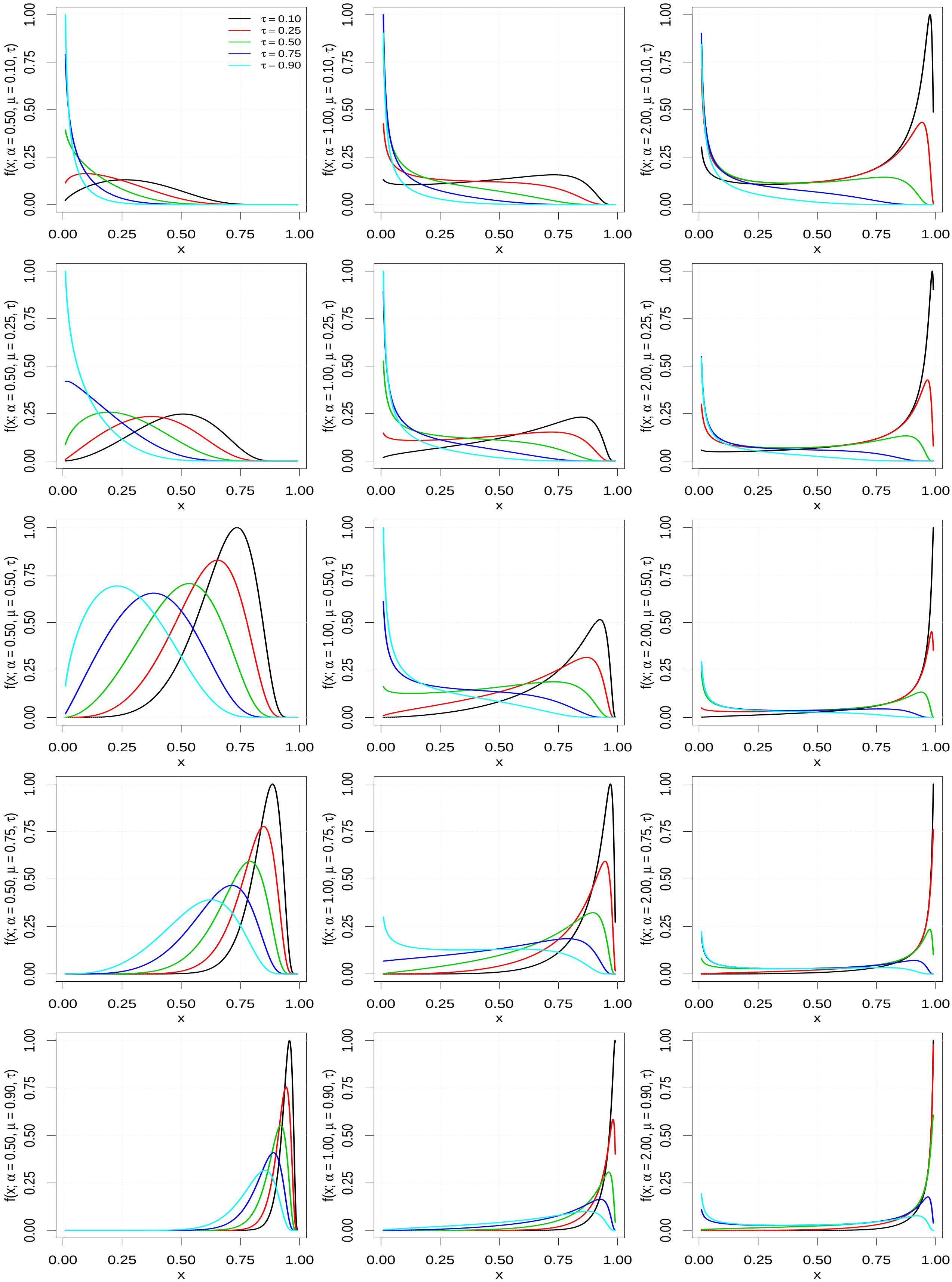

Figure 1. PDFs of the unit UBS distribution for the indicated values of μ, α and τ.

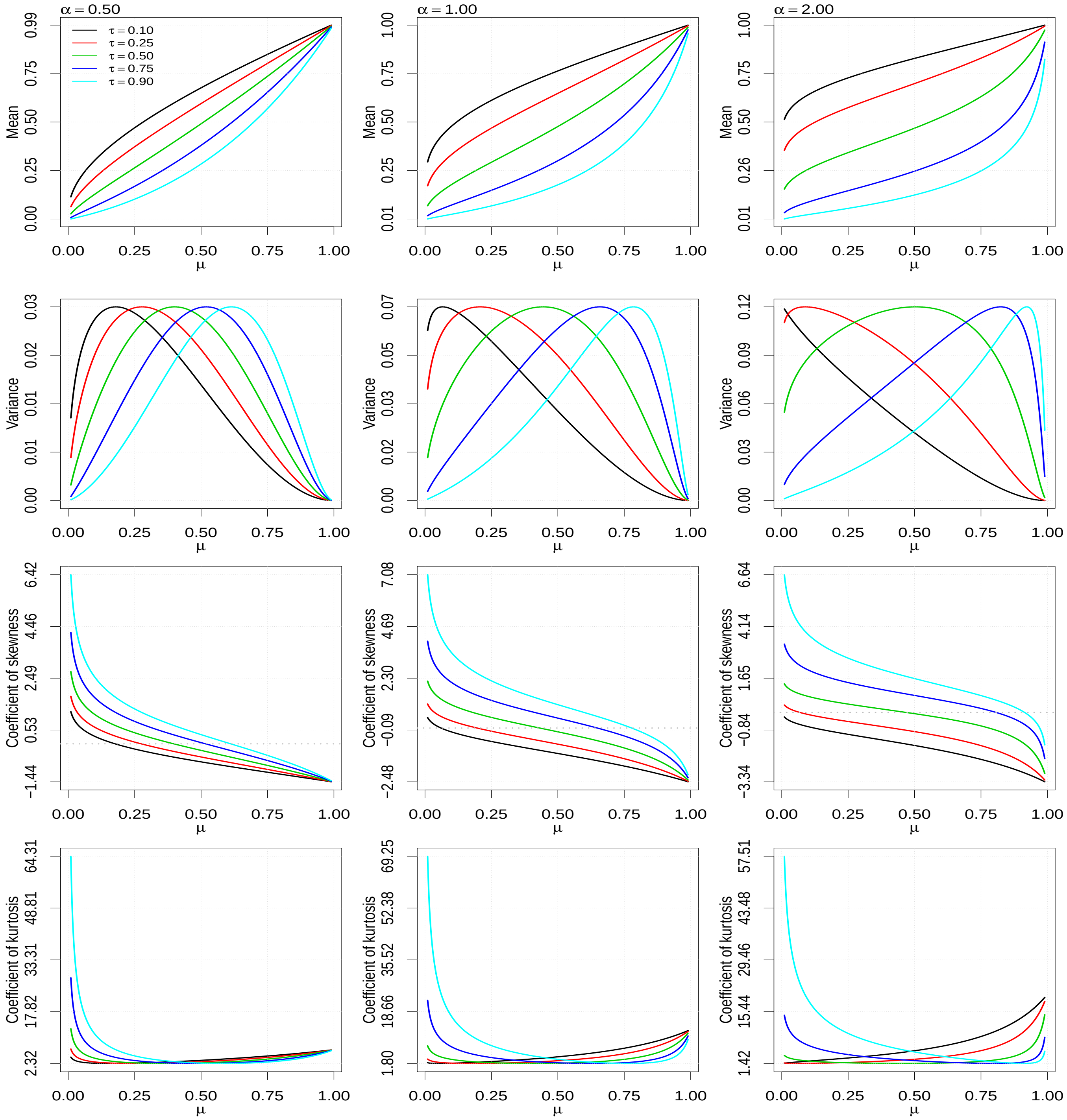

Figure 2. Behaviors of the mean, variance, coefficient of skewness and coefficient of kurtosis for the indicated values of α and τ in function of μ.

The standard ML method can be used to estimate the UBS parameters by maximizing the corresponding log-likelihood function. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a realization of the random sample $\boldsymbol{X} = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent and identically distributed random variables according to (1). Using (5), the corresponding log-likelihood function, with $\boldsymbol{\theta} = (\mu, \alpha)$, is written as

$$\ell(\boldsymbol{\theta};\boldsymbol{x},\tau) \propto -n\log[\alpha\log(\mu)h(\alpha,\tau)]+\frac{n}{\alpha^2}+\frac{1}{2\alpha^2}\sum_{i=1}^{n}\frac{\log(x_i)}{\log(\mu)h(\alpha,\tau)}$$
$$+\sum_{i=1}^{n}\log\left\{\left[-\frac{\log(\mu)h(\alpha,\tau)}{\log(x_i)}\right]^{\frac{1}{2}}+\left[-\frac{\log(\mu)h(\alpha,\tau)}{\log(x_i)}\right]^{\frac{3}{2}}\right\}+\frac{1}{2\alpha^2}\sum_{i=1}^{n}\frac{\log(\mu)h(\alpha,\tau)}{\log(x_i)}.$$

Defining $\boldsymbol{U}(\boldsymbol{\theta};\boldsymbol{x},\tau)=(U_\mu,U_\alpha)$, where $U_\mu=\partial\ell(\boldsymbol{\theta};\boldsymbol{x},\tau)/\partial\mu$, $U_\alpha=\partial\ell(\boldsymbol{\theta};\boldsymbol{x},\tau)/\partial\alpha$, and $h'(\alpha,\tau)=\partial h(\alpha,\tau)/\partial\alpha$, we get the associated score vector with coordinates stated as

$$U_\mu = -\frac{n}{\mu\log(\mu)}-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\frac{\log(x_i)}{\log(\mu)^2h(\alpha,\tau)}+\frac{1}{2\alpha^2}\sum_{i=1}^{n}\frac{h(\alpha,\tau)}{\mu\log(x_i)} \quad (7)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\frac{\log(x_i)-3\log(\mu)h(\alpha,\tau)}{\log(\mu)\mu[\log(\mu)h(\alpha,\tau)-\log(x_i)]},$$
$$U_\alpha = -\frac{2n}{\alpha^3}-\frac{n}{\alpha h(\alpha,\tau)}[h(\alpha,\tau)+\alpha h'(\alpha,\tau)]-\frac{[2h(\alpha,\tau)+h'(\alpha,\tau)\alpha]}{2\alpha^3\log(\mu)h(\alpha,\tau)^2}\sum_{i=1}^{n}\log(x_i) \quad (8)$$
$$+\frac{h'(\alpha,\tau)}{2h(\alpha,\tau)}\sum_{i=1}^{n}\frac{\log(x_i)-3\log(\mu)h(\alpha,\tau)}{\log(x_i)-\log(\mu)h(\alpha,\tau)}-\frac{\log(\mu)[2h(\alpha,\tau)-h'(\alpha,\tau)\alpha]}{2\alpha^3}\sum_{i=1}^{n}\frac{1}{\log(x_i)}.$$

By solving numerically the system of non-linear equations in μ and α, formed by the coordinates of $\boldsymbol{U}(\boldsymbol{\theta};\boldsymbol{x},\tau)$ defined in (7) and (8), we have the ML estimate of $\widehat{\boldsymbol{\theta}}$. The asymptotic standard errors (SEs) of the corresponding estimators are obtained by inverting the Hessian matrix of the log-likelihood function. This Hessian matrix is generated by taking the second derivative of (7), with respect to μ and α.

For $\tau=0.5$, we have that

$$\widehat{\alpha}=\frac{1}{n\log(\widehat{\mu})}\sqrt{n\log(\widehat{\mu})\left[\sum_{i=1}^{n}\log(x_i)+\log^2(\widehat{\mu})\sum_{i=1}^{n}\frac{1}{\log(x_i)}-2n\log(\widehat{\mu})\right]}=\xi(\widehat{\mu}),$$

while $\widehat{\mu}$ must be found numerically taking $\alpha=\xi(\widehat{\mu})$ in the expression defined in (7).

3. The UBS Quantile Regression Model

In this section, considering the parameterized PDF stated in (5), we formulate the UBS quantile regression model.

Let $X_1,\ldots,X_n$ be n independent random variables, where each X_i, for $i=1,\ldots,n$, follows the PDF given in (5) with unknown quantile parameter μ_i, unknown shape parameter α, and $\tau\in(0,1)$ is assumed as fixed, that is, $X_i\sim\mathrm{UBS}(\mu_i,\alpha;\tau)$. Here, the UBS quantile regression model is defined imposing that the quantile μ_i of X_i satisfies the functional relation expressed by

$$g(\mu_i)=\boldsymbol{z}_i^\top\boldsymbol{\delta},\qquad i=1,\ldots,n, \quad (9)$$

where $\boldsymbol{\delta}=(\delta_0,\ldots,\delta_{p-1})^\top$ is a p-dimensional vector of unknown regression coefficients, with $p<n$ and n being the sample size, $\boldsymbol{z}_i=(1,z_{i1},\ldots,z_{i(p-1)})^\top$ denotes the observations on p known covariates, and g is given as in (4). There are several possibilities for the link function g stated in (9). For instance, the most useful well-known link functions are:

(i) [Logit] $g(\mu_i)=\log[\mu_i/(1-\mu_i)]$;
(ii) [Probit] $g(\mu_i)=\Phi^{-1}(\mu_i)$;
(iii) [Complementary log-log] $g(\mu_i)=\log[-\log(1-\mu_i)]$;
(iv) [Log-log] $g(\mu_i)=\log[-\log(\mu_i)]$; and
(v) [Cauchit] $g(\mu_i)=\tan[\pi(\mu-0.5)]$;

which are the inverse CDF of the logistic, standard normal, minimum extreme-value, maximum extreme-value and Cauchy distributions, respectively.

Let $X_1, \ldots, X_n$ be n independent random variables such that $X_i \sim \text{UBS}(\mu_i, \alpha; \tau)$ denotes a UBS distributed random variable and consider that

$$\mu_i = \frac{\exp(z_i^\top \delta)}{1 + \exp(z_i^\top \delta)}, \qquad i = 1, \ldots, n,$$

to ensure that the predicted quantiles lie within the unit interval.

For fixed $\tau \in (0,1)$, let now $\boldsymbol{\theta} = (\delta^\top, \alpha)^\top$ be the vector of $p+1$ unknown parameters to be estimated. The ML estimate $\widehat{\boldsymbol{\theta}} = (\widehat{\delta}^\top, \widehat{\alpha})^\top$ of $\boldsymbol{\theta}$ is obtained by the maximizing the log-likelihood function stated in (7). It is not possible to derive analytical solution for the ML estimate $\widehat{\boldsymbol{\theta}}$ so that it must be calculated numerically using some optimization algorithm such as Newton-Raphson and quasi-Newton. Based on [15], we suggest to use, as an initial guess for δ, the ordinary least squares estimate of this parameter vector obtained from the linear regression of the transformed responses $g(x_1), \ldots, g(x_n)$ on $\boldsymbol{Z}$, that is, $\widehat{\delta} = (\boldsymbol{Z}^\top \boldsymbol{Z})^{-1} Z^\top \boldsymbol{w}$, where $\boldsymbol{w} = (g(x_1), \ldots, g(x_n))^\top$.

Under mild regularity conditions and when the sample size n is large, the asymptotic distribution of the ML estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\delta}^\top, \widehat{\alpha})^\top$ is approximately multivariate normal (of dimension $p+1$) with mean vector $\boldsymbol{\theta} = (\delta^\top, \alpha)^\top$ and variance covariance matrix $\boldsymbol{K}^{-1}(\boldsymbol{\theta})$, where

$$\boldsymbol{K}(\boldsymbol{\theta}) = \mathrm{E}\left[-\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^\top}\right]$$

is the expected Fisher information matrix. Note that there is no closed form expression for the matrix $\boldsymbol{K}(\boldsymbol{\theta})$. Nevertheless, as shown in [52], the estimated observed Fisher information matrix

$$\boldsymbol{J}(\widehat{\boldsymbol{\theta}}) = -\left.\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^\top}\right|_{\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}}$$

is a consistent estimator of the expected Fisher information matrix $\boldsymbol{K}(\boldsymbol{\theta})$. Therefore, for large n, we can replace $\boldsymbol{K}(\boldsymbol{\theta})$ by $\boldsymbol{J}(\widehat{\boldsymbol{\theta}})$.

Let θ_r be the r-th component of $\boldsymbol{\theta}$. The asymptotic $100 \times (1-\gamma)\%$ confidence interval for θ_r is given by $[\widehat{\theta}_r \pm z_{\gamma/2}\, \widehat{\text{SE}}(\widehat{\theta}_r)]$, with $r = 1, \ldots, p+1$, where $z_{\gamma/2}$ is the $100 \times \gamma/2$ upper quantile of the standard normal distribution and $\widehat{\text{SE}}(\widehat{\theta}_r)$ is the estimated asymptotic SE of $\widehat{\theta}_r$. Note that $\widehat{\text{SE}}(\widehat{\theta}_r)$ is the square root of the r-th diagonal element of the matrix $\boldsymbol{J}^{-1}(\widehat{\boldsymbol{\theta}})$.

Given n pairs of observations $(x_i, \boldsymbol{z}_i)$, the parameter estimates of $\delta = (\delta_0, \ldots, \delta_{p-1})^\top$ and α can be obtained directly through a library of the `R` software named `unitBSQuantReg` by means of its function `unitBSQuantReg(formula, tau, data, link,...)`. For model checking, the function `hnp(object, nsim = 99, halfnormal = TRUE, plot = TRUE, level = 0.95, resid.type = c("cox-snell", "quantile"))` is available, which provides half-normal probability plots with simulated envelopes. A version of the `unitBSQuantReg` package is available at https://github.com/AndrMenezes/unitBSQuantReg (accessed on 9 April 2021).

Due to the direct interpretation of the parameters in terms of odds, in this paper, we consider only the logit link. When μ_i is the τ-th quantile, for $0 < \tau < 1$, the interpretations are straightforward. In addition, a strictly positive link function relating the shape parameter β with covariates $\boldsymbol{w}_i$, not necessarily equal to $\boldsymbol{x}_i$, can be considered. Of course, other link functions might be explored.

4. A Monte Carlo Simulation Study

In this section, we present the results of the Monte Carlo simulations used to assess the empirical bias and root mean-squared error (RMSE) of the ML estimators of the UBS quantile regression parameters. We also present the coverage probability (CP) of the 95% confidence interval ($\text{CP}_{95\%}$) based on asymptotic normality of the ML estimators. We consider sample sizes $n = 20, 50, 100, 200, 300$; $\tau = 0.10, 0.25, 0.50, 0.75, 0.90$; $\alpha = 0.5, 1.0, 2.0$, and two regression frameworks formulated as

(i) $\text{logit}(\mu_i) = \delta_0 + \delta_1 z_{i1}$ for $\delta_0 = 1.0$, $\delta_1 = 2.0$, with $z_{i1} \sim \text{N}(0, 1)$; and
(ii) $\text{logit}(\mu_i) = \delta_0 + \delta_1 z_{i1} + \delta_2 z_{i2}$ for $\delta_0 = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 2.0$, $z_{i1} \sim \text{N}(0, 1)$, with $z_{i2} \sim \text{N}(0, 1)$.

For each combination of (n, τ, α) and both regression frameworks (the covariate(s) remained constant throughout the simulations), $B = 5000$ pseudo-random samples were simulated using the `SAS Data-Step`, while parameter estimates were obtained by the quasi-Newton method in `SAS PROC NLMIXED`. The values of the response variable, given n, τ, α and the covariate(s), are generated from the quantile function stated as

$$q = \exp\left(-2\beta/\left\{2 + \left[\alpha\Phi^{-1}(1-U)\right]^2 - \alpha\Phi^{-1}(1-U)\sqrt{4 + [\alpha\Phi^{-1}(1-U)]^2}\right\}\right),$$

where $U \sim \text{U}(0, 1)$ and $\beta = \log(\mu)h(\alpha, \tau)$ for $h(\alpha, \tau)$ defined in (4).

The empirical bias, RMSE and CP are calculated, respectively, by

(i) $\text{Bias}(\widehat{\varrho}) = (1/B)\sum_{i=1}^{B}(\widehat{\varrho}_i - \varrho)$;
(ii) $\text{RMSE}(\widehat{\varrho}_i) = [(1/B)\sum_{i=1}^{B}(\widehat{\varrho}_i - \varrho)^2]^{1/2}$; and
(iii) $\text{CP}_{95\%}(\widehat{\varrho}_i) = (1/B)\sum_{i=1}^{B} I[\widehat{\varrho}_i \pm 1.96\text{SE}(\widehat{\varrho}_i)]$, where $\varrho = \alpha, \delta_0, \delta_1$, or δ_2, I is the indicator function, and $\text{SE}(\widehat{\varrho}_i)$ is the corresponding estimated SE.

Tables 1–3 and Tables 4–6 report the results for the first and second regression structures respectively. These tables reveal a low bias in the estimation of α and δ for all scenarios. The empirical RMSE is also low and quickly tends to zero as the sample size increases. Higher values of bias and RMSE are observed as the quantiles are distant from $\tau = 0.5$, either from the left or right. Still, for all scenarios, the CP tends to the nominal confidence coefficient as the sample size increases. In summary, the simulations reveal that the UBS quantitative regression model has its parameters well estimated according to the metrics considered and can be an alternative to other models available in the literature.

Table 1. Empirical bias, RMSE and 95% CP for the true values $\delta_0 = 1.0$, $\delta_1 = 2.0$ and $\alpha = 0.5$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	**Bias**			**RMSE**			**CP$_{95\%}$**		
		δ_0	δ_1	α	δ_0	δ_1	α	δ_0	δ_1	α
0.10	20	0.050	−0.007	−0.034	0.202	0.174	0.085	0.887	0.916	0.853
	50	0.021	−0.003	−0.013	0.123	0.103	0.051	0.927	0.938	0.911
	100	0.010	−0.002	−0.006	0.087	0.073	0.036	0.936	0.940	0.932
	200	0.005	−0.001	−0.003	0.061	0.051	0.025	0.943	0.944	0.941
	300	0.003	−0.000	−0.002	0.050	0.041	0.020	0.948	0.947	0.946
0.25	20	0.026	−0.002	−0.034	0.168	0.173	0.086	0.903	0.918	0.854
	50	0.011	−0.002	−0.013	0.103	0.102	0.051	0.935	0.940	0.911
	100	0.005	−0.001	−0.006	0.073	0.072	0.036	0.940	0.941	0.931
	200	0.003	−0.000	−0.003	0.051	0.050	0.025	0.944	0.945	0.940
	300	0.002	−0.000	−0.002	0.042	0.041	0.020	0.950	0.947	0.946
0.50	20	−0.003	0.004	−0.034	0.152	0.172	0.086	0.918	0.918	0.856
	50	−0.000	0.001	−0.013	0.094	0.102	0.051	0.938	0.941	0.912
	100	−0.000	−0.000	−0.006	0.066	0.072	0.036	0.944	0.943	0.932
	200	0.000	0.000	−0.003	0.047	0.050	0.025	0.946	0.946	0.940
	300	0.000	0.000	−0.002	0.038	0.040	0.020	0.949	0.949	0.947
0.75	20	−0.033	0.011	−0.034	0.171	0.174	0.085	0.914	0.920	0.855
	50	−0.012	0.003	−0.013	0.104	0.103	0.051	0.934	0.944	0.914
	100	−0.006	0.001	−0.006	0.073	0.072	0.036	0.944	0.942	0.933
	200	−0.003	0.001	−0.003	0.052	0.050	0.025	0.946	0.946	0.940
	300	−0.002	0.001	−0.002	0.042	0.041	0.020	0.948	0.949	0.947
0.90	20	−0.058	0.018	−0.033	0.207	0.177	0.085	0.904	0.920	0.857
	50	−0.022	0.006	−0.013	0.125	0.104	0.051	0.930	0.944	0.915
	100	−0.011	0.002	−0.006	0.087	0.073	0.036	0.941	0.942	0.935
	200	−0.005	0.002	−0.003	0.061	0.051	0.025	0.947	0.947	0.939
	300	−0.003	0.001	−0.002	0.049	0.041	0.020	0.950	0.949	0.945

Table 2. Empirical bias, RMSE and 95% CP for the true values $\delta_0 = 1.0$, $\delta_1 = 2.0$ and $\alpha = 1.0$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	Bias			RMSE			$CP_{95\%}$		
		δ_0	δ_1	α	δ_0	δ_1	α	δ_0	δ_1	α
0.10	20	0.094	−0.007	−0.073	0.367	0.320	0.173	0.884	0.918	0.846
	50	0.038	−0.005	−0.029	0.222	0.189	0.103	0.925	0.939	0.908
	100	0.018	−0.003	−0.014	0.157	0.133	0.072	0.936	0.941	0.930
	200	0.009	−0.001	−0.007	0.110	0.092	0.050	0.944	0.946	0.939
	300	0.006	−0.000	−0.004	0.089	0.075	0.041	0.948	0.947	0.945
0.25	20	0.050	0.001	−0.073	0.312	0.318	0.173	0.901	0.918	0.847
	50	0.021	−0.002	−0.029	0.190	0.188	0.103	0.934	0.940	0.908
	100	0.010	−0.002	−0.014	0.134	0.131	0.072	0.938	0.942	0.930
	200	0.005	0.000	−0.007	0.095	0.092	0.050	0.944	0.946	0.939
	300	0.003	0.000	−0.004	0.077	0.074	0.041	0.949	0.948	0.945
0.50	20	−0.010	0.013	−0.073	0.283	0.320	0.173	0.916	0.922	0.847
	50	−0.003	0.003	−0.029	0.172	0.188	0.103	0.939	0.942	0.909
	100	−0.001	0.000	−0.014	0.121	0.131	0.072	0.945	0.944	0.930
	200	−0.000	0.001	−0.007	0.085	0.091	0.050	0.945	0.946	0.940
	300	−0.000	0.001	−0.004	0.069	0.074	0.041	0.948	0.948	0.946
0.75	20	−0.074	0.029	−0.072	0.325	0.327	0.173	0.915	0.923	0.849
	50	−0.028	0.009	−0.029	0.194	0.190	0.103	0.936	0.944	0.910
	100	−0.013	0.003	−0.014	0.135	0.132	0.072	0.945	0.944	0.931
	200	−0.006	0.002	−0.007	0.095	0.092	0.050	0.946	0.947	0.939
	300	−0.004	0.002	−0.004	0.077	0.074	0.041	0.950	0.948	0.946
0.90	20	−0.122	0.043	−0.071	0.390	0.335	0.171	0.908	0.924	0.850
	50	−0.047	0.014	−0.028	0.228	0.192	0.102	0.932	0.946	0.912
	100	−0.023	0.005	−0.014	0.158	0.133	0.071	0.943	0.944	0.933
	200	−0.010	0.003	−0.007	0.111	0.093	0.050	0.947	0.947	0.939
	300	−0.007	0.002	−0.004	0.089	0.075	0.040	0.949	0.950	0.945

Table 3. Empirical bias, RMSE and 95% CP for the true values $\delta_0 = 1.0$, $\delta_1 = 2.0$ and $\alpha = 2.0$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	Bias			RMSE			$CP_{95\%}$		
		δ_0	δ_1	α	δ_0	δ_1	α	δ_0	δ_1	α
0.10	20	0.152	−0.004	−0.162	0.566	0.511	0.358	0.887	0.921	0.831
	50	0.061	−0.005	−0.064	0.333	0.292	0.209	0.928	0.941	0.901
	100	0.029	−0.004	−0.031	0.233	0.201	0.145	0.938	0.941	0.928
	200	0.014	−0.001	−0.015	0.163	0.139	0.101	0.945	0.947	0.938
	300	0.009	0.000	−0.009	0.132	0.112	0.081	0.950	0.947	0.944
0.25	20	0.095	0.006	−0.162	0.502	0.512	0.358	0.897	0.922	0.833
	50	0.039	−0.002	−0.064	0.297	0.291	0.210	0.930	0.942	0.901
	100	0.019	−0.002	−0.031	0.208	0.200	0.145	0.941	0.943	0.928
	200	0.010	0.000	−0.015	0.146	0.138	0.101	0.946	0.947	0.938
	300	0.006	0.001	−0.009	0.118	0.111	0.082	0.948	0.946	0.944
0.50	20	−0.026	0.032	−0.160	0.450	0.522	0.358	0.917	0.924	0.832
	50	−0.008	0.007	−0.063	0.262	0.292	0.210	0.939	0.943	0.903
	100	−0.004	0.002	−0.030	0.181	0.199	0.145	0.945	0.944	0.929
	200	−0.001	0.002	−0.014	0.127	0.138	0.101	0.946	0.946	0.938
	300	−0.001	0.002	−0.009	0.102	0.111	0.082	0.949	0.948	0.945

Table 3. *Cont.*

τ	n	Bias			RMSE			$CP_{95\%}$		
		δ_0	δ_1	α	δ_0	δ_1	α	δ_0	δ_1	α
0.75	20	−0.157	0.069	−0.158	0.558	0.549	0.356	0.922	0.928	0.833
	50	−0.058	0.020	−0.062	0.311	0.299	0.208	0.941	0.945	0.905
	100	−0.028	0.008	−0.030	0.212	0.202	0.144	0.947	0.945	0.929
	200	−0.013	0.005	−0.014	0.148	0.139	0.101	0.946	0.946	0.938
	300	−0.009	0.004	−0.009	0.119	0.111	0.081	0.949	0.949	0.946
0.90	20	−0.227	0.087	−0.160	0.653	0.569	0.354	0.918	0.930	0.833
	50	−0.083	0.024	−0.065	0.353	0.303	0.208	0.938	0.947	0.904
	100	−0.040	0.007	−0.033	0.238	0.203	0.144	0.945	0.946	0.928
	200	−0.020	0.003	−0.018	0.166	0.140	0.101	0.947	0.946	0.935
	300	−0.014	0.001	−0.013	0.133	0.112	0.081	0.948	0.950	0.942

Table 4. Empirical bias, RMSE and 95% CP for the true values $\delta_0 = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 2.0$ and $\alpha = 0.5$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	Bias				RMSE				$CP_{95\%}$			
		δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α
0.10	20	0.066	−0.006	−0.010	−0.046	0.213	0.160	0.176	0.091	0.876	0.916	0.912	0.813
	50	0.027	−0.002	−0.005	−0.019	0.128	0.095	0.103	0.052	0.920	0.930	0.938	0.901
	100	0.013	−0.001	−0.002	−0.009	0.090	0.065	0.073	0.036	0.933	0.939	0.941	0.924
	200	0.006	−0.000	−0.001	−0.005	0.063	0.045	0.051	0.025	0.939	0.946	0.945	0.934
	300	0.004	−0.001	−0.000	−0.003	0.052	0.036	0.041	0.021	0.941	0.948	0.945	0.940
0.25	20	0.033	−0.003	−0.003	−0.047	0.176	0.160	0.175	0.091	0.898	0.916	0.915	0.812
	50	0.014	−0.001	−0.002	−0.019	0.107	0.095	0.102	0.053	0.928	0.930	0.940	0.899
	100	0.007	−0.000	−0.001	−0.009	0.075	0.065	0.072	0.037	0.940	0.939	0.942	0.924
	200	0.003	−0.000	−0.000	−0.005	0.053	0.045	0.050	0.026	0.941	0.946	0.946	0.934
	300	0.002	−0.000	0.000	−0.003	0.043	0.036	0.041	0.021	0.941	0.947	0.944	0.939
0.50	20	−0.008	0.001	0.005	−0.047	0.158	0.161	0.175	0.092	0.914	0.917	0.919	0.813
	50	−0.003	0.001	0.001	−0.019	0.097	0.095	0.102	0.053	0.937	0.930	0.941	0.895
	100	−0.001	0.000	0.001	−0.009	0.069	0.065	0.072	0.037	0.940	0.938	0.943	0.922
	200	−0.001	0.000	0.001	−0.005	0.048	0.045	0.050	0.026	0.945	0.947	0.947	0.931
	300	−0.001	−0.000	0.001	−0.003	0.039	0.036	0.041	0.021	0.944	0.947	0.944	0.939
0.75	20	−0.049	0.006	0.015	−0.046	0.180	0.162	0.177	0.091	0.899	0.917	0.920	0.813
	50	−0.020	0.002	0.005	−0.019	0.109	0.095	0.103	0.053	0.928	0.931	0.942	0.894
	100	−0.010	0.001	0.002	−0.009	0.076	0.065	0.072	0.037	0.937	0.938	0.942	0.920
	200	−0.005	0.001	0.001	−0.005	0.053	0.045	0.050	0.026	0.941	0.948	0.946	0.932
	300	−0.004	0.000	0.001	−0.003	0.043	0.036	0.041	0.021	0.945	0.947	0.945	0.938
0.90	20	−0.085	0.010	0.024	−0.046	0.222	0.164	0.181	0.090	0.879	0.917	0.919	0.815
	50	−0.034	0.004	0.008	−0.019	0.131	0.096	0.105	0.053	0.921	0.931	0.940	0.894
	100	−0.017	0.002	0.004	−0.009	0.091	0.065	0.073	0.036	0.934	0.937	0.943	0.920
	200	−0.009	0.001	0.002	−0.005	0.064	0.045	0.051	0.025	0.940	0.948	0.947	0.933
	300	−0.006	0.000	0.002	−0.003	0.051	0.036	0.041	0.021	0.945	0.948	0.945	0.937

Table 5. Empirical bias, RMSE and 95% CP for the true values: $\delta_0 = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 2.0$ and $\alpha = 1.0$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	Bias				RMSE				$CP_{95\%}$			
		δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α
0.10	20	0.124	−0.009	−0.013	−0.101	0.388	0.297	0.326	0.185	0.871	0.916	0.914	0.799
	50	0.050	−0.003	−0.007	−0.041	0.231	0.175	0.190	0.106	0.919	0.932	0.942	0.894
	100	0.025	−0.001	−0.003	−0.020	0.162	0.119	0.133	0.073	0.933	0.939	0.941	0.922
	200	0.012	−0.001	−0.001	−0.010	0.113	0.082	0.092	0.051	0.940	0.949	0.943	0.933
	300	0.008	−0.001	−0.000	−0.007	0.093	0.066	0.075	0.041	0.943	0.948	0.945	0.939

Table 5. *Cont.*

τ	n	Bias				RMSE				$CP_{95\%}$			
		δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α
0.25	20	0.063	−0.004	−0.001	−0.101	0.327	0.298	0.325	0.186	0.895	0.919	0.916	0.799
	50	0.026	−0.001	−0.002	−0.041	0.198	0.175	0.189	0.106	0.927	0.933	0.942	0.893
	100	0.013	−0.000	−0.001	−0.020	0.139	0.119	0.132	0.073	0.936	0.939	0.944	0.921
	200	0.006	−0.000	−0.000	−0.010	0.097	0.082	0.092	0.051	0.940	0.949	0.945	0.932
	300	0.004	−0.000	0.000	−0.007	0.080	0.066	0.074	0.042	0.942	0.947	0.945	0.938
0.50	20	−0.022	0.005	0.017	−0.101	0.296	0.301	0.328	0.187	0.915	0.919	0.921	0.801
	50	−0.008	0.002	0.005	−0.041	0.179	0.175	0.189	0.107	0.937	0.933	0.943	0.891
	100	−0.004	0.001	0.002	−0.020	0.126	0.119	0.131	0.074	0.940	0.939	0.944	0.920
	200	−0.003	0.001	0.002	−0.010	0.088	0.082	0.091	0.052	0.944	0.949	0.946	0.930
	300	−0.002	0.000	0.001	−0.007	0.072	0.066	0.074	0.042	0.946	0.948	0.945	0.938
0.75	20	−0.110	0.016	0.038	−0.100	0.349	0.306	0.337	0.186	0.902	0.920	0.921	0.801
	50	−0.044	0.006	0.013	−0.041	0.204	0.177	0.191	0.107	0.931	0.933	0.944	0.889
	100	−0.021	0.003	0.006	−0.020	0.142	0.119	0.132	0.074	0.934	0.939	0.944	0.918
	200	−0.012	0.002	0.003	−0.010	0.099	0.082	0.092	0.051	0.943	0.949	0.947	0.930
	300	−0.008	0.001	0.003	−0.007	0.080	0.066	0.074	0.042	0.946	0.949	0.945	0.937
0.90	20	−0.177	0.025	0.056	−0.098	0.425	0.311	0.347	0.184	0.885	0.922	0.921	0.801
	50	−0.070	0.010	0.019	−0.040	0.242	0.178	0.195	0.106	0.924	0.934	0.942	0.889
	100	−0.034	0.005	0.009	−0.020	0.166	0.120	0.134	0.073	0.933	0.939	0.945	0.918
	200	−0.018	0.002	0.005	−0.010	0.115	0.082	0.093	0.051	0.942	0.949	0.948	0.929
	300	−0.012	0.001	0.004	−0.007	0.093	0.066	0.075	0.041	0.944	0.949	0.945	0.937

Table 6. Empirical bias, RMSE and 95% CP for the true values $\delta_0 = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 2.0$ and $\alpha = 2.0$ of the indicated quantile level (τ), sample size (n) and parameter ($\delta_0, \delta_1, \alpha$) with simulated data.

τ	n	Bias				RMSE				$CP_{95\%}$			
		δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α	δ_0	δ_1	δ_2	α
0.10	20	0.203	−0.012	−0.009	−0.225	0.613	0.489	0.534	0.391	0.867	0.917	0.913	0.775
	50	0.080	−0.003	−0.008	−0.090	0.352	0.275	0.297	0.217	0.919	0.937	0.941	0.884
	100	0.040	−0.001	−0.004	−0.044	0.243	0.183	0.203	0.148	0.929	0.942	0.944	0.916
	200	0.019	−0.000	−0.001	−0.022	0.169	0.123	0.139	0.103	0.939	0.951	0.944	0.929
	300	0.012	−0.001	−0.000	−0.015	0.137	0.100	0.112	0.083	0.942	0.950	0.946	0.937
0.25	20	0.123	−0.005	0.006	−0.224	0.540	0.493	0.537	0.392	0.887	0.917	0.916	0.776
	50	0.050	−0.000	−0.002	−0.090	0.314	0.276	0.296	0.218	0.926	0.938	0.942	0.884
	100	0.025	0.000	−0.002	−0.044	0.218	0.182	0.202	0.149	0.933	0.942	0.945	0.917
	200	0.011	0.000	0.000	−0.022	0.151	0.123	0.139	0.103	0.942	0.951	0.945	0.929
	300	0.007	−0.000	0.001	−0.015	0.123	0.100	0.111	0.084	0.942	0.950	0.946	0.937
0.50	20	−0.049	0.013	0.043	−0.223	0.488	0.506	0.552	0.392	0.919	0.919	0.921	0.779
	50	−0.017	0.006	0.012	−0.090	0.277	0.278	0.298	0.220	0.938	0.939	0.944	0.882
	100	−0.008	0.004	0.005	−0.045	0.190	0.183	0.201	0.150	0.941	0.942	0.947	0.916
	200	−0.005	0.002	0.003	−0.023	0.132	0.123	0.138	0.104	0.945	0.951	0.947	0.927
	300	−0.004	0.001	0.003	−0.015	0.107	0.100	0.111	0.084	0.946	0.951	0.947	0.936
0.75	20	−0.234	0.038	0.093	−0.220	0.624	0.528	0.585	0.389	0.909	0.923	0.923	0.779
	50	−0.089	0.015	0.029	−0.089	0.335	0.282	0.306	0.219	0.933	0.940	0.945	0.879
	100	−0.043	0.007	0.012	−0.045	0.225	0.184	0.203	0.149	0.936	0.942	0.947	0.913
	200	−0.023	0.004	0.007	−0.023	0.155	0.124	0.139	0.103	0.943	0.951	0.947	0.927
	300	−0.016	0.002	0.005	−0.015	0.124	0.100	0.112	0.083	0.944	0.950	0.948	0.936
0.90	20	−0.330	0.051	0.119	−0.222	0.739	0.541	0.607	0.387	0.902	0.925	0.927	0.776
	50	−0.126	0.018	0.035	−0.094	0.384	0.285	0.311	0.219	0.928	0.940	0.946	0.876
	100	−0.063	0.008	0.013	−0.050	0.255	0.186	0.205	0.149	0.935	0.941	0.947	0.910
	200	−0.033	0.003	0.005	−0.028	0.174	0.124	0.140	0.104	0.940	0.951	0.948	0.923
	300	−0.023	0.001	0.003	−0.020	0.139	0.101	0.113	0.084	0.942	0.947	0.950	0.932

5. Empirical Application with Data from Political Sciences

In this and the following section, for illustrative purposes, we apply the UBS quantile regression model in the analysis of two real data sets. In addition to the UBS model, we also consider quantile regression models based on the KUMA, LEEG, UBUR, UCHE, UHN, unit logistic (ULOG) and UWEI, distributions. The PDF, CDF and QF for these distributions are presented in Appendix A.

The objective of these sections is not to present all numerous approaches to variable selection, regression diagnostics, link function selection or parameter interpretation, but rather suggest the use of the UBS quantile regression as an alternative to other models available in the literature.

In this first application, the response variable indicates the proportion of voters in the 2014 Brazil presidential election, received by the elected president, Dilma Rousseff, in 74 municipalities of the state of Sergipe. This state is located in the Northeast Region of Brazil and is formed by 75 municipalities. The data set is available in the library `baquantreg` [53] from `R` and was considered the human development index (HDI) in 2010 as covariate. We call this set "vote data" and assume the regression structure for μ_i formulated as $\text{logit}(\mu_i) = \delta_0 + \delta_1\,\text{HDI}_i$, for $i = 1, \ldots, 74$.

Table 7 reports the ML estimates and SEs for the indicated models. Note the difference between the estimates of δ_0 and δ_1 in the models. The rate of change in the conditional quantile, expressed by the estimated regression coefficients, is illustrated in Figure 3. Observe how the rate of change in the proportion of votes depends on the quantile. As also shown in Figure 3, the first plot on the left, a higher HDI indicates less proportion of votes received by the elected president.

Table 7. ML estimates and SEs of the indicated model, parameter and quantile level (τ) with vote data.

Model	Parameter	Estimate					SE				
		τ Level					τ Level				
		0.10	**0.25**	**0.50**	**0.75**	**0.90**	**0.10**	**0.25**	**0.50**	**0.75**	**0.90**
	α	12.204	12.209	12.215	12.222	12.228	1.152	1.152	1.152	1.151	1.151
KUMA	δ_0	1.717	2.153	2.649	3.158	3.629	0.462	0.529	0.612	0.702	0.786
	δ_1	−2.081	−2.425	−2.862	−3.352	−3.835	0.764	0.877	1.015	1.161	1.296
	α	18.176	18.183	18.190	18.198	18.207	1.825	1.825	1.825	1.826	1.826
LEEG	δ_0	2.155	2.506	2.937	3.481	4.197	0.415	0.450	0.496	0.562	0.660
	δ_1	−2.708	−3.005	−3.395	−3.924	−4.662	0.677	0.738	0.816	0.919	1.062
	α	0.279	0.279	0.279	0.279	0.279	0.023	0.023	0.023	0.023	0.023
UBS	δ_0	2.457	2.596	2.760	2.925	3.077	0.670	0.645	0.623	0.605	0.592
	δ_1	−3.281	−3.168	−3.066	−2.978	−2.916	1.123	1.081	1.043	1.012	0.989
	α	40.007	40.004	40.085	40.663	40.661	11.258	11.249	11.280	11.539	11.538
UBURR	δ_0	1.907	2.294	2.784	3.398	4.068	0.423	0.484	0.570	0.671	0.788
	δ_1	−2.368	−2.733	−3.184	−3.763	−4.390	0.692	0.797	0.938	1.102	1.289
	α	3.712	3.712	3.712	3.712	3.712	0.335	0.335	0.335	0.334	0.334
UCHEN	δ_0	2.246	2.392	2.578	2.803	3.057	0.665	0.661	0.650	0.634	0.619
	δ_1	−2.939	−2.908	−2.848	−2.766	−2.683	1.113	1.106	1.087	1.057	1.026
UHN	δ_0	0.742	1.672	2.422	2.956	3.314	1.408	1.408	1.408	1.408	1.408
	δ_1	−3.000	−3.000	−3.000	−3.000	−3.000	2.353	2.353	2.353	2.353	2.353
	α	5.304	5.304	5.304	5.305	5.304	0.512	0.512	0.512	0.512	0.512
ULOG	δ_0	2.506	2.713	2.921	3.127	3.335	0.566	0.564	0.563	0.563	0.564
	δ_1	−3.338	−3.338	−3.338	−3.338	−3.338	0.939	0.939	0.939	0.939	0.939
	α	3.768	3.768	3.767	3.767	3.767	0.324	0.324	0.324	0.324	0.324
UWEI	δ_0	2.302	2.414	2.573	2.781	3.027	0.697	0.676	0.652	0.629	0.610
	δ_1	−3.032	−2.940	−2.835	−2.729	−2.637	1.169	1.132	1.091	1.049	1.013

Where we recall that KUMA: Kumaraswamy, LEEG: log-extended exponential-geometric, UBS: unit Birnbaum–Saunders, UBUR: unit Burr XII, UCHE: unit Chen, UHN: unit half-normal, ULOG: unit logistic, UWEI: unit Weibull.

ML estimates for the UBS model parameters can be obtained from the `unitBSQuantReg` package using the instructions:

```
install.packages(‘‘remotes’’)
remotes::install_github(‘‘AndrMenezes/unitBSQuantReg’’)
library(unitBSQuantReg)
data(BrazilElection2014, package = ‘‘baquantreg’’)
ind     <- which(BrazilElection2014$UF_name == ‘‘SE’’)
data.se <- BrazilElection2014[ind, c(‘‘percVotes", ‘‘HDI’’)]
taus    <- c(0.10, 0.25, 0.50, 0.75, 0.90)
fits    <- lapply(taus, function(TAU)
unitBSQuantReg(percVotes ~ HDI, data = data.se, tau = TAU))
lapply(fits, coef)
```

For the other models, the ML estimates were obtained from the `unitquantreg` package, which is under development.

Table 8 presents the values of the Akaike information criterion used to compare various competing models. These values indicate that the UBS quantile regression model is the best one. This better performance, of course, is not a rule and depends on the data under analysis, but this fact highlights the importance of our new proposal for such data. In order to assess the goodness of fit of the UBS quantile regression model to the vote data, in Figure 4 are shown the half-normal plots with simulated envelopes for the Cox–Snell and randomized quantile residuals. These figures indicate a good fit of the UBS quantile regression model to the τ-th proportion of votes, for $\tau \in (0.10, 0.25, 0.50, 0.75, 0.90)$.

Table 8. Values of the Akaike information criterion for the indicated model and quantile level (τ) with vote data.

Model	τ Level				
	0.10	**0.25**	**0.50**	**0.75**	**0.90**
KUMA	−186.6585	−186.7418	−186.8450	−186.9566	−187.0622
LEEG	−181.1746	−181.2213	−181.2757	−181.3365	−181.3965
UBS	−188.7764	−188.7825	−188.7883	−188.7932	−188.7969
UBUR	−180.7540	−180.7524	−180.7513	−180.7531	−180.7528
UCHE	−178.9413	−178.9489	−178.9513	−178.9480	−178.9416
UHN	−112.5021	−112.5021	−112.5021	−112.5021	−112.5021
ULOG	−186.1657	−186.1657	−186.1657	−186.1657	−186.1657
UWEI	−179.3758	−179.3678	−179.3581	−179.3479	−179.3385

Where we recall that KUMA: Kumaraswamy, LEEG: log-extended exponential-geometric, UBS: unit Birnbaum–Saunders, UBUR: unit Burr XII, UCHE: unit Chen, UHN: unit half-normal, ULOG: unit logistic, UWEI: unit Weibull.

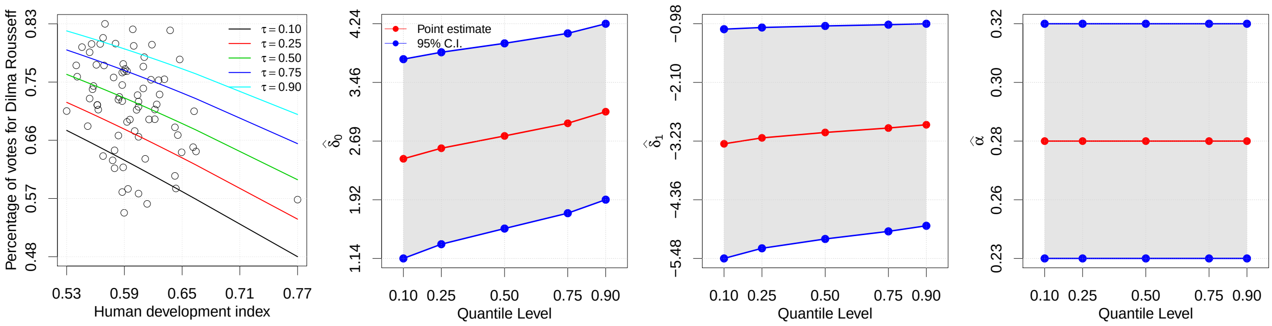

Figure 3. Quantile regression fit plot (**left**) and estimated quantile process plots for δ_0 (**center left**), δ_1 (**center right**) and α (**right**) with vote data.

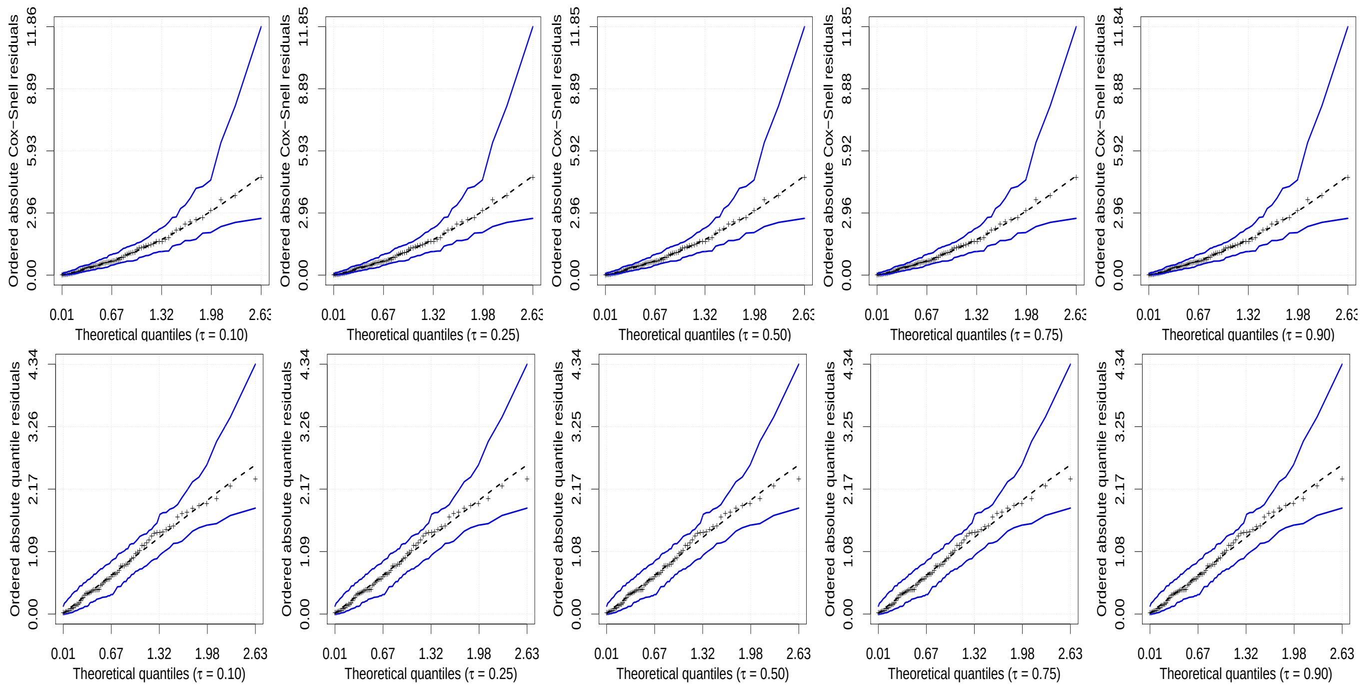

Figure 4. Half-normal plots with envelopes of Cox–Snell (first row) and randomized quantile (second row) residuals for the indicated τ quantile level with vote data.

6. Empirical Application with Data from Sports Medicine

In this second application, we consider a data set taken from [54], also available at http://www.leg.ufpr.br/doku.php/publications:papercompanions:multquasibeta (accessed on 9 April 2021). These data are related to the body fat percentage, which was measured at five regions: android, arms, gynoid, legs, and trunk. The fat percentages at android, arms, gynoid, legs, and trunk correspond to the five response variables considered in the original study where these data were collected. The data set contains 298 observations and the covariates are: age (in years), body mass index (in kg/m^2), gender (female or male) and IPAQ (sedentary (S), insufficiently active (I), or active (A)). As described in [55], the IPAQ is a questionnaire that allows the estimation of weekly time spent on physical activities of moderate and strong intensity, in different contexts of daily life, such as: housework, leisure, transportation, and work, as well as the time spent in passive activities performed on the seating position.

We analyze the response variable body fat percentage at arms. We call this set as "arm data". The results of the analyses for the other four responses are available under request from the authors. For the response variable analyzed here, we assume μ_i is given by $\text{logit}(\mu_i) = \delta_0 + \delta_1 z_{i1} + \delta_2 z_{i2} + \delta_3 z_{i3} + \delta_4 z_{i4} + \delta_5 z_{i5}$, for $i = 1, \ldots, 298$, where z_{i1}: age; z_{i2}: body mass index; z_{i3}: 0 for female, 1 for male; z_{i4}: 0 for IPAQ = S, 1 for IPAQ = I; and z_{i5}: 0 for IPAQ = S, 1 for IPAQ = A.

Table 9 reports the ML estimates and SEs for the indicated models. Note the difference between the estimates of δ_j, for $j = 0, 1, \ldots, 5$, in the models. The rate of change in the conditional quantile, expressed by the estimated regression coefficients, is illustrated in Figure 5. Note how the rate of change in body fat percentage at arms depends on the quantile.

Table 10 presents the values of the Akaike information criterion used to compare various models. These values indicate that the UBS quantile regression model is the best one. In order to assess if this model fits the data well, in Figure 6 are shown the half-normal plots with simulated envelopes for the residuals. This figure displays a good fit of the UBS quantile regression model to the τ-th percentage of body fat at arms, for $\tau \in (0.10, 0.25, 0.50, 0.75, 0.90)$.

Table 9. ML estimates and SEs of the indicated model, parameter and quantile level (τ) with arm data.

Model	Parameter	Estimate					SE				
		τ Level					τ Level				
		0.10	0.25	0.50	0.75	0.90	0.10	0.25	0.50	0.75	0.90
	α	4.705	4.715	4.724	4.731	4.736	0.210	0.210	0.210	0.210	0.210
	δ_0	−3.071	−2.894	−2.754	−2.658	−2.599	0.135	0.140	0.147	0.156	0.163
	δ_1	0.004	0.004	0.004	0.004	0.005	0.001	0.001	0.001	0.001	0.001
KUMA	δ_2	7.291	7.721	8.226	8.743	9.210	0.551	0.577	0.609	0.644	0.676
	δ_3	−0.730	−0.771	−0.820	−0.870	−0.915	0.033	0.035	0.037	0.039	0.041
	δ_4	−0.072	−0.074	−0.076	−0.078	−0.078	0.047	0.050	0.054	0.057	0.060
	δ_5	−0.196	−0.206	−0.216	−0.227	−0.236	0.047	0.050	0.053	0.056	0.060
	α	7.587	7.609	7.629	7.648	7.671	0.375	0.375	0.375	0.375	0.374
	δ_0	−3.159	−3.056	−2.968	−2.897	−2.850	0.142	0.147	0.154	0.163	0.175
	δ_1	0.004	0.005	0.005	0.005	0.006	0.001	0.001	0.001	0.001	0.001
LEEG	δ_2	8.331	8.746	9.269	9.934	10.785	0.650	0.676	0.711	0.757	0.817
	δ_3	−0.870	−0.909	−0.957	−1.020	−1.100	0.035	0.037	0.039	0.041	0.045
	δ_4	−0.114	−0.122	−0.131	−0.143	−0.159	0.051	0.053	0.056	0.060	0.065
	δ_5	−0.234	−0.244	−0.257	−0.272	−0.292	0.047	0.050	0.052	0.055	0.060
	α	0.160	0.160	0.160	0.159	0.159	0.007	0.007	0.007	0.007	0.007
	δ_0	−3.394	−3.105	−2.801	−2.522	−2.289	0.160	0.150	0.141	0.134	0.129
	δ_1	0.005	0.005	0.004	0.004	0.004	0.001	0.001	0.001	0.001	0.001
UBS	δ_2	9.525	9.045	8.581	8.161	7.820	0.705	0.666	0.629	0.596	0.572
	δ_3	−1.001	−0.947	−0.896	−0.850	−0.813	0.040	0.038	0.035	0.033	0.032
	δ_4	−0.127	−0.122	−0.116	−0.111	−0.108	0.053	0.051	0.048	0.046	0.044
	δ_5	−0.271	−0.256	−0.244	−0.232	−0.223	0.052	0.049	0.047	0.044	0.042
	α	0.453	0.457	0.459	0.461	0.463	0.043	0.043	0.043	0.043	0.043
	δ_0	−1.485	−1.109	−0.782	−0.514	−0.283	0.496	0.257	0.098	0.030	0.109
	δ_1	0.002	0.001	0.000	−0.000	−0.000	0.006	0.003	0.001	0.000	0.001
UBUR	δ_2	−0.302	−0.003	0.045	−0.008	−0.095	2.675	1.346	0.507	0.056	0.482
	δ_3	−1.035	−0.495	−0.182	0.019	0.168	0.225	0.102	0.037	0.019	0.044
	δ_4	0.114	0.057	0.021	−0.002	−0.020	0.186	0.095	0.036	0.004	0.034
	δ_5	−0.122	−0.072	−0.030	0.003	0.030	0.207	0.105	0.040	0.005	0.038
	α	1.799	1.805	1.809	1.805	1.783	0.039	0.039	0.039	0.040	0.041
	δ_0	−3.572	−3.503	−3.396	−3.214	−2.872	0.189	0.203	0.223	0.252	0.293
	δ_1	0.005	0.006	0.007	0.007	0.008	0.002	0.002	0.002	0.002	0.002
UCHEN	δ_2	9.700	10.393	11.374	12.547	13.522	0.825	0.880	0.959	1.068	1.218
	δ_3	−0.962	−1.033	−1.138	−1.265	−1.371	0.048	0.052	0.056	0.064	0.073
	δ_4	−0.120	−0.128	−0.138	−0.144	−0.139	0.075	0.080	0.087	0.096	0.103
	δ_5	−0.329	−0.353	−0.387	−0.428	−0.460	0.073	0.078	0.085	0.094	0.104
	δ_0	−4.854	−3.923	−3.173	−2.639	−2.282	0.350	0.350	0.350	0.350	0.350
	δ_1	0.004	0.004	0.004	0.004	0.004	0.003	0.003	0.003	0.003	0.003
UHN	δ_2	8.754	8.754	8.755	8.754	8.755	1.533	1.534	1.533	1.531	1.533
	δ_3	−0.884	−0.884	−0.884	−0.884	−0.884	0.086	0.086	0.086	0.086	0.086
	δ_4	−0.108	−0.108	−0.108	−0.108	−0.108	0.126	0.126	0.126	0.126	0.126
	δ_5	−0.242	−0.242	−0.242	−0.242	−0.242	0.122	0.122	0.122	0.122	0.122
	α	5.821	5.821	5.821	5.821	5.821	0.282	0.282	0.282	0.282	0.282
	δ_0	−3.265	−3.076	−2.888	−2.699	−2.510	0.146	0.145	0.145	0.145	0.145
	δ_1	0.005	0.005	0.005	0.005	0.005	0.001	0.001	0.001	0.001	0.001
ULOG	δ_2	8.878	8.878	8.878	8.878	8.878	0.663	0.661	0.662	0.661	0.661
	δ_3	−0.932	−0.932	−0.932	−0.932	−0.932	0.036	0.036	0.036	0.036	0.036
	δ_4	−0.122	−0.122	−0.122	−0.122	−0.122	0.051	0.051	0.051	0.051	0.051
	δ_5	−0.239	−0.239	−0.239	−0.239	−0.239	0.049	0.049	0.049	0.049	0.049
	α	5.976	5.980	5.985	5.992	6.000	0.243	0.243	0.243	0.243	0.243
	δ_0	−3.214	−2.960	−2.640	−2.272	−1.895	0.168	0.159	0.148	0.138	0.130
	δ_1	0.006	0.005	0.005	0.005	0.004	0.001	0.001	0.001	0.001	0.001
UWEI	δ_2	8.592	8.201	7.729	7.218	6.735	0.729	0.691	0.647	0.600	0.556
	δ_3	−0.966	−0.919	−0.863	−0.803	−0.746	0.043	0.040	0.037	0.035	0.032
	δ_4	−0.131	−0.127	−0.121	−0.115	−0.109	0.055	0.053	0.050	0.047	0.044
	δ_5	−0.355	−0.339	−0.319	−0.298	−0.277	0.055	0.053	0.050	0.047	0.043

Where we recall that KUMA: Kumaraswamy, LEEG: log-extended exponential-geometric, UBS: unit Birnbaum–Saunders, UBUR: unit Burr XII, UCHE: unit Chen, UHN: unit half-normal, ULOG: unit logistic, UWEI: unit Weibull.

Table 10. Values of the Akaike information criterion for the indicated model and quantile level (τ) with arm data.

Model	**τ Level**				
	0.10	**0.25**	**0.50**	**0.75**	**0.90**
KUMA	−839.660	−842.224	−845.030	−847.657	−849.813
LEEG	−846.433	−849.810	−853.814	−858.493	−863.789
UBS	−905.942	−907.590	−909.332	−910.964	−912.330
UBUR	−466.813	−467.464	−467.982	−468.339	−468.586
UCHEN	−690.483	−691.734	−691.722	−687.523	−676.318
UHN	−546.283	−546.283	−546.283	−546.283	−546.283
ULOG	−888.401	−888.401	−888.401	−888.401	−888.401
UWEI	−838.151	−839.364	−840.932	−842.757	−844.603

Where we recall that KUMA: Kumaraswamy, LEEG: log-extended exponential-geometric, UBS: unit Birnbaum–Saunders, UBUR: unit Burr XII, UCHE: unit Chen, UHN: unit half-normal, ULOG: unit logistic, UWEI: unit Weibull.

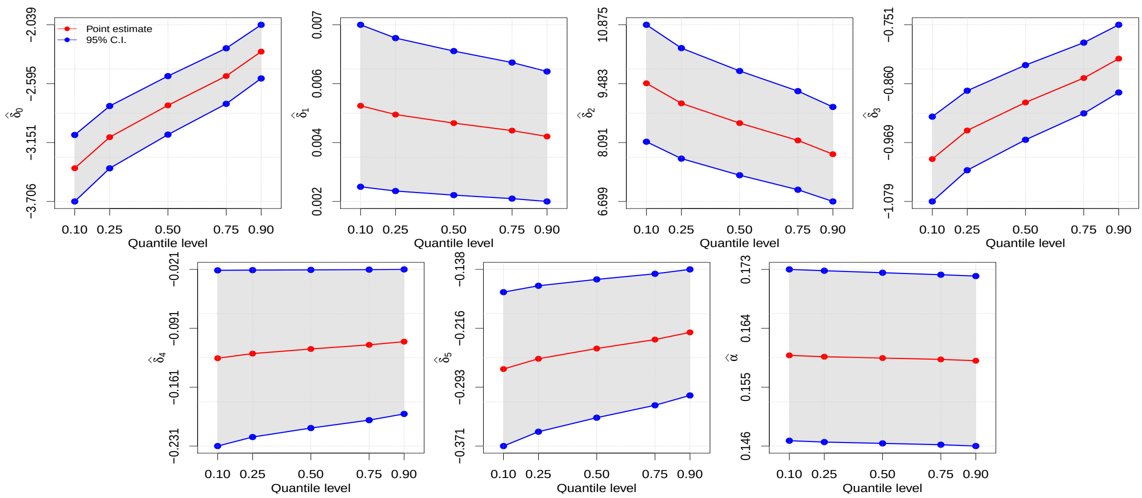

Figure 5. Estimated quantile process plot for the indicated δ_j, with $j = 0, 1, \ldots, 5$, and α (second row right) with arm data.

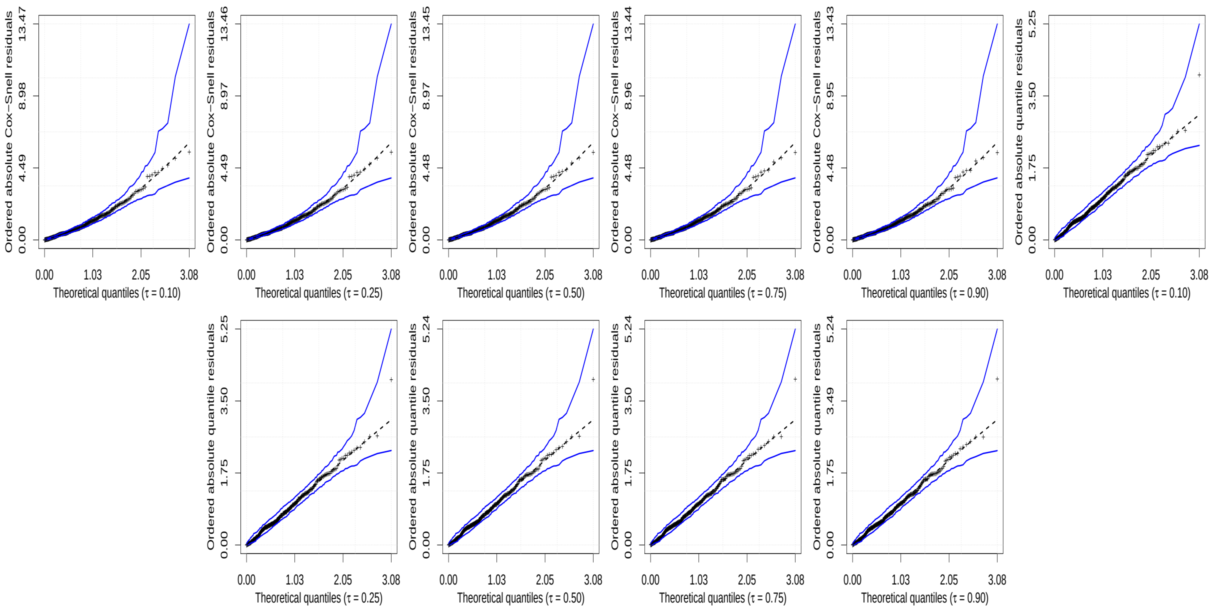

Figure 6. Half-normal plots with envelopes of Cox–Snell (first row) and randomized quantile (second row) residuals for the indicated τ quantile level with arm data.

7. Conclusions, Limitations and Future Investigation

Although the quantile regression methodology appeared in 1978 [34], few works consider it from a strictly parametric point of view. Some of these recently published works used distributions for responses restricted to the zero-one interval.

In this paper, we extended the unit Birnbaum–Saunders distribution by parameterizing its scale parameter in terms of a quantile that can depend on covariates. This extension employed a quantile parameterization that permitted us to state a setting mimicking the generalized linear models, giving wide flexibility in the formulation.

We estimated the model parameter with the maximum likelihood method and used Cox–Snell and randomized quantile residuals to assess the adequacy of our formulation to the data, as well as the Akaike information criterion for model selection.

We conducted a Monte Carlo simulation study to empirically evaluate the statistical performance of the maximum likelihood estimators of the unit Birnbaum–Saunders quantile regression parameters. We also studied coverage probabilities of the confidence intervals of the corresponding parameters based on the asymptotic normality of these estimators. This simulation study reported the good statistical performance of such estimators.

Two data analyses were conducted related to political sciences and sports medicine with seven competing models to the unit Birnbaum–Saunders quantile regression. Both of these analyses reported a suitable performance of the new regression quantile model and superior to all of the competing models, giving evidence that the unit Birnbaum–Saunders distribution is an excellent alternative for quantile modeling and for dealing with bounded data into the unit interval. Such results reported that the unit Birnbaum–Saunders quantile regression model may be a new setting for analyzing this type of data. The new approach may be a good addition to the tools of statisticians and diverse practitioners interested in the modeling of quantiles.

A limitation of our approach is that the explanatory variables may condition quantiles and also the shape parameter. Thus, the effect of this parameter is a topic to be studied in a future work based on the line presented in [56,57] when jointly modeling two parameters. Other topics of future research are related to studying how multivariate, spatial, temporal structures could be added into the quantile regression framework [58–61]. In addition, Tobit and Cobb-Douglas type frameworks may be analyzed in the thematic of this investigation [62,63]. Moreover, censored observation might be studied in the present context [64].

The authors are working on these and other issues associated with the present investigation and the corresponding findings are expected to be reported in future works.

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Appendix A. Other Distributions for Quantile Regression

• The KUMA distribution [65] is obtained from the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{EE}(\alpha, \beta)$ denotes an exponentiated-exponential distributed random variable. The corresponding PDF, CDF and QF are written, respectively, as

$$\begin{aligned} f(x;\alpha,\beta) &= \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\alpha,\beta) &= 1-(1-x^{\alpha})^{\beta}, \\ Q(\tau;\alpha,\beta) &= \left[1-(1-\tau)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}, \quad 0<\tau<1. \end{aligned} \tag{A1}$$

From (A1), the parameter β can be expressed as $\beta = g^{-1}(\mu) = \log(1-\tau)/\log(1-\mu^{\alpha})$.

• The LEEG distribution [28] is generated from the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{EEG}(\alpha, \beta)$ denotes an extended exponential-geometric distributed random variable. The corresponding PDF, CDF and QF are stated, respectively, as

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\alpha(1+\beta)x^{\alpha-1}}{(1+\beta x^{\alpha})^2}, \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\alpha,\beta) &= \frac{(1+\beta)x^{\alpha}}{1+\beta x^{\alpha}}, \\ Q(\tau;\alpha,\beta) &= \left(\frac{\tau}{1+\beta-\beta\tau}\right)^{\frac{1}{\alpha}}, \quad 0<\tau<1. \end{aligned} \tag{A2}$$

From (A2), the parameter β can be expressed as $\beta = g^{-1}(\mu) = (\tau-\mu^{\alpha})/\mu^{\alpha}(1-\tau)$.

• The UBUR distribution [33] is reached from the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{BUR}(\alpha, \beta)$ denotes a Burr XII distributed random variable. The corresponding PDF, CDF and QF are given, respectively, by

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\alpha\beta}{x}[-\log(x)]^{\beta-1}\left\{1+[-\log(x)]^{\beta}\right\}^{-\alpha-1}, \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\alpha,\beta) &= \left\{1+[-\log(x)]^{\beta}\right\}^{-\alpha}, \\ Q(\tau;\alpha,\beta) &= \exp\left[-\left(\tau^{-\frac{1}{\alpha}}-1\right)\right]^{\frac{1}{\beta}}, \quad 0<\tau<1. \end{aligned} \tag{A3}$$

From (A3), the parameter β can be expressed as $\beta = g^{-1}(\mu) = \log(\tau^{-1/\alpha}-1)/\log[-\log(\mu)]$.

• The UCHE distribution [31] is get from the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{CHE}(\alpha, \beta)$ denotes a Chen distributed random variable. The corresponding PDF, CDF and QF are expressed, respectively, as

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\alpha\beta}{x}[-\log(x)]^{\alpha-1}\exp\{[-\log(x)]^{\alpha}\}\exp\{\beta\{1-\exp[(-\log(x))^{\alpha}]\}\}, \\ F(x;\alpha,\beta) &= \exp\{\beta\{1-\exp[(-\log(x))^{\alpha}]\}\}, \quad 0<x<1, \alpha>0, \beta>0, \\ Q(\tau;\alpha,\beta) &= \exp\left\{-\left[\log\left(1-\frac{\log(\tau)}{\beta}\right)\right]^{\frac{1}{\alpha}}\right\}, \quad 0<\tau<1. \end{aligned} \tag{A4}$$

From (A4), the parameter β can be stated as $\beta = g^{-1}(\mu) = \log(\tau)/(1-\exp\{[-\log(x)]^{\alpha}\})$.

• The UHN distribution [32] is obtained from the transformation $X = Y/(1+Y)$, where $Y \sim \mathrm{HN}(\beta)$ denotes a half-normal distributed random variable. The corresponding PDF, CDF and QF are formulated, respectively, as

$$\begin{aligned} f(x;\beta) &= \frac{2}{\beta(1-x)^2}\phi\left(\frac{x}{\beta(1-x)}\right), \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\beta) &= 2\Phi\left(\frac{x}{\beta(1-x)}\right)-1, \\ Q(\tau;\beta) &= \frac{\beta\Phi^{-1}\left(\frac{\tau+1}{2}\right)}{1+\beta\Phi^{-1}\left(\frac{\tau+1}{2}\right)}, \quad 0<\tau<1, \end{aligned} \tag{A5}$$

where ϕ is the PDF of the standard normal distribution. From (A5), the parameter β can be expressed as $\beta = g^{-1}(\mu) = \mu/[(1-\mu)\Phi^{-1}(\tau+1/2)]$.

• The ULOG distribution [29] is get from the transformation $X = \log[Y/(1+Y)]$, where $Y \sim \mathrm{LOG}(\alpha,\beta)$ denotes a logistic distributed random variable. The corresponding PDF, CDF and QF are established, respectively, by

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\beta\alpha x^{\alpha-1}(1-x)^{\alpha-1}}{[\beta x^\alpha+(1-x)^\alpha]^2}, \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\alpha,\beta) &= \frac{1}{[1+\beta[(1-x)/x]^\alpha]}, \\ Q(\tau;\alpha,\beta) &= \frac{1}{1+[(1-\tau)/(\beta\tau)]^{\frac{1}{\alpha}}}, \quad 0<\tau<1. \end{aligned} \tag{A6}$$

From (A6), the parameter β can be expressed as $\beta = g^{-1}(\mu) = [(1-\tau)/\tau][\mu/(1-\mu)]^\alpha$.

• The UWEI distribution [8] is generated from the transformation $X = \exp(-Y)$, where $Y \sim \mathrm{WEI}(\alpha,\beta)$ denotes a Weibull distributed random variable. The corresponding PDF, CDF and QF are written, respectively, as

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\alpha\beta}{x}[-\log(x)]^{\alpha-1}\exp\{-\beta[-\log(x)]^\alpha\}, \quad 0<x<1, \alpha>0, \beta>0, \\ F(x;\alpha,\beta) &= \exp\{-\beta[-\log(x)]^\alpha\}, \\ Q(\tau;\alpha,\beta) &= \exp\left\{-\left[-\frac{\log(\tau)}{\beta}\right]^{\frac{1}{\alpha}}\right\}, \quad 0<\tau<1. \end{aligned} \tag{A7}$$

From (A7) the parameter β can be expressed as $\beta = g^{-1}(\mu) = -\log(\tau)/[-\log(\mu)]^\alpha$.

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